10

Mathematics

Learner's Module
Unit 1

This book was collaboratively developed and reviewed by educators from public and private schools, colleges, and/or universities. We encourage teachers and other education stakeholders to email their feedback, comments, and recommendations to the Department of Education at action@deped.gov.ph.

We value your feedback and recommendations.

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Learner's Module
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Introduction

This material is written in support of the K to 12 Basic Education Program to ensure attainment of standards expected of students.

In the design of this Grade 10 materials, it underwent different processes - development by writers composed of classroom teachers, school heads, supervisors, specialists from the Department and other institutions; validation by experts, academicians, and practitioners; revision; content review and language editing by members of Quality Circle Reviewers; and finalization with the guidance of the consultants.

There are eight (8) modules in this material.

Module 1 – Sequences
Module 2 – Polynomials and Polynomial Equations
Module 3 – Polynomial Functions
Module 4 – Circles
Module 5 – Plane Coordinate Geometry
Module 6 – Permutations and Combinations
Module 7 – Probability of Compound Events
Module 8 – Measures of Position

With the different activities provided in every module, may you find this material engaging and challenging as it develops your critical-thinking and problem-solving skills.

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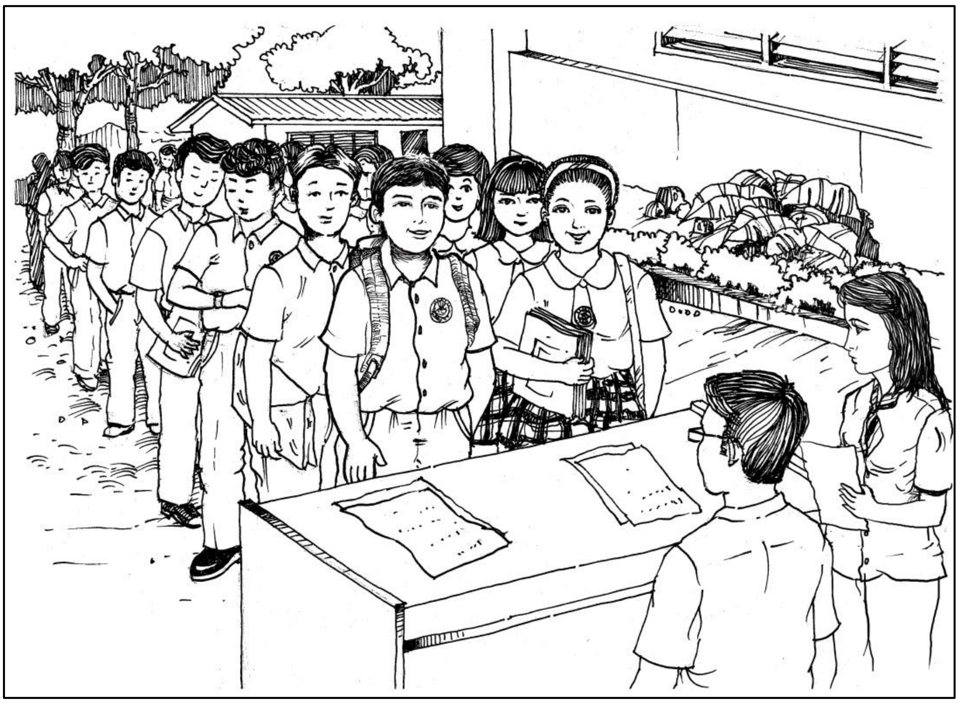
Unit 1

1 SEQUENCES

I. INTRODUCTION



“Kilos Kabataan”

In her first public address, the principal mentioned about the success of the recent “Brigada Eskwela.” Because of this success, the principal challenged the students, especially the Grade 9 and Grade 10 students, to extend the same service to their community by having a one-Saturday community clean-up which the principal called “Kilos Kabataan Project.” Volunteers have to sign up until 5 p.m. for the project. Accepting the principal’s challenge, 10 students immediately signed up for the clean-up. After 10 minutes, there were already 15 who had signed up. After 10 more minutes, there were 20, then 25, 30, and so on. Amazed by the students’ response to the challenge, the principal became confident that the youth could be mobilized to create positive change.

The above scenario illustrates a sequence. In this learning module, you will know more about sequences, and how the concept of a sequence is utilized in our daily lives.

II. LESSONS AND COVERAGE

In this module, you will learn more about sequences when you take the following lessons:

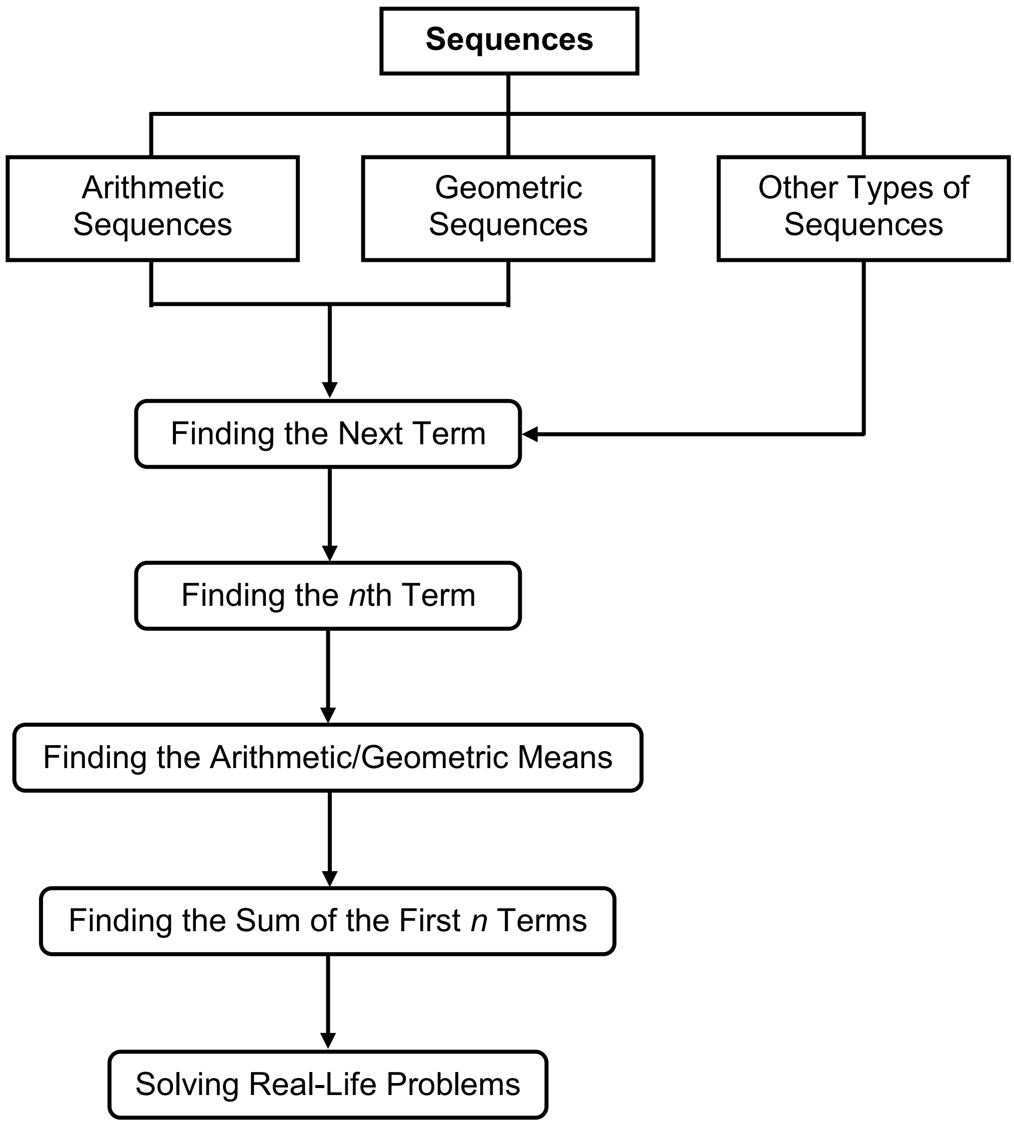
Lesson 1 – Arithmetic Sequences

Lesson 2 – Geometric and Other Sequences

In these lessons you will learn to:

Lesson 1	• generate and describe patterns • find the next few terms of a sequence • find the general or *n*th term of a sequence • illustrate an arithmetic sequence • determine the *n*th term of a given arithmetic sequence • find the arithmetic means between terms of an arithmetic sequence • determine the sum of the first *n* terms of a given arithmetic sequence • solve problems involving arithmetic sequence
Lesson 2	• illustrate a geometric sequence • differentiate a geometric sequence from an arithmetic sequence • determine the *n*th term of a given geometric sequence • find the geometric means between terms of a geometric sequence • determine the sum of the first *n* terms of a geometric sequence • determine the sum of the first *n* terms of an infinite geometric sequence • illustrate other types of sequences like harmonic sequence and Fibonacci sequence • solve problems involving geometric sequence

Module Map


Sequences
Arithmetic Sequences
Geometric Sequences
Other Types of Sequences
Finding the Next Term
Finding the *n*th Term
Finding the Arithmetic/Geometric Means
Finding the Sum of the First *n* Terms
Solving Real-Life Problems

III. PRE-ASSESSMENT

Part 1

Find out how much you already know about the topics in this module. Choose the letter of the best answer. Take note of the items that you were not able to answer correctly and find the right answer as you go through this module.

1. What is the next term in the geometric sequence 4, –12, 36?

   A. –42　　B. –54　　C. –72　　D. –108

2. Find the common difference in the arithmetic sequence $3, \frac{13}{4}, \frac{7}{2}, \frac{15}{4}, \ldots$

   A. $\frac{1}{4}$　　B. $\frac{3}{4}$　　C. $\frac{5}{2}$　　D. 4

3. Which set of numbers is an example of a harmonic sequence?

   A. $\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2},$　　C. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$

   B. $\frac{1}{2}, -1, 2, -4$　　D. $2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}$

4. What is the sum of all the odd integers between 8 and 26?

   A. 153　　B. 151　　C. 149　　D. 148

5. If three arithmetic means are inserted between 11 and 39, find the second arithmetic mean.

   A. 18　　B. 25　　C. 32　　D. 46

6. If three geometric means are inserted between 1 and 256, find the third geometric mean.

   A. 64　　B. 32　　C. 16　　D. 4

7. What is the next term in the harmonic sequence $\frac{1}{11}, \frac{1}{15}, \frac{1}{19}, \frac{1}{23}, \ldots$?

   A. 27　　B. 25　　C. $\frac{1}{25}$　　D. $\frac{1}{27}$

8. Which term of the arithmetic sequence $4, 1, -2, -5, \ldots$ is -29?
A. 9th term B. 10th term C. 11th term D. 12th term

9. What is the 6th term of the geometric sequence $\frac{2}{25}, \frac{2}{5}, 2, 10, \ldots$?
A. 25 B. 250 C. 1250 D. 2500

10. The first term of an arithmetic sequence is 2 while the 18th term is 87. Find the common difference of the sequence.
A. 7 B. 6 C. 5 D. 3

11. What is the next term in the Fibonacci sequence 1, 1, 2, 3, 5, 8, ...?
A. 13 B. 16 C. 19 D. 20

12. Find the sum of the geometric sequence where the first term is 3, the last term is 46 875, and the common ratio is 5.
A. 58 593 B. 58 594 C. 58 595 D. 58 596

13. Find the eighth term of a geometric sequence where the third term is 27 and the common ratio is 3.
A. 2187 B. 6561 C. 19 683 D. 59 049

14. Which of the following is the sum of all the multiples of 3 from 15 to 48?
A. 315 B. 360 C. 378 D. 396

15. What is the 7th term of the sequence whose nth term is $a_n = \frac{n^2-1}{n^2+1}$?
A. $\frac{24}{25}$ B. $\frac{23}{25}$ C. $\frac{47}{50}$ D. $\frac{49}{50}$

16. What is the nth term of the arithmetic sequence 7, 9, 11, 13, 15, 17, . .?
A. $3n+4$ B. $4n+3$ C. $n+2$ D. $2n+5$

17. What is the nth term of the harmonic sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$?
A. $\frac{1}{n+1}$ B. $\frac{1}{n^2+1}$ C. $\frac{1}{2n}$ D. $\frac{1}{4n-2}$

18. Find p so that the numbers $7p+2,\ 5p+12,\ 2p-1, \ldots$ form an arithmetic sequence.
A. –8 B. –5 C. –13 D. –23

19. What is the sum of the infinite geometric series $\frac{3}{4}-\frac{9}{16}+\frac{27}{64}-\frac{81}{256}+\ldots$?
A. 3 B. 1 C. $\frac{3}{4}$ D. $\frac{3}{7}$

20. Find k so that the numbers $2k+1$, $3k+4$, and $7k+6$ form a geometric sequence.
A. 2; -1 B. -2; 1 C. 2; 1 D. -2; -1

21. Glenn bought a car for Php600,000. The yearly depreciation of his car is 10% of its value at the start of the year. What is its value after 4 years?
A. Php437,400 B. Php438,000 C. Php393,660 D. Php378,000

22. During a free-fall, a skydiver jumps 16 feet, 48 feet, and 80 feet on the first, second, and third fall, respectively. If he continues to jump at this rate, how many feet will he have jumped during the tenth fall?
A. 304 B. 336 C. 314 928 D. 944 784

23. Twelve days before Valentine's Day, Carl decided to give Nicole flowers according to the Fibonacci sequence. On the first day, he sent one red rose, on the second day, two red roses, and so on. How many roses did Nicole receive during the tenth day?
A. 10 B. 55 C. 89 D. 144

24. A new square is formed by joining the midpoints of the consecutive sides of a square 8 inches on a side. If the process is continued until there are already six squares, find the sum of the areas of all squares in square inches.
A. 96 B. 112 C. 124 D. 126

25. In President Sergio Osmeña High School, suspension of classes is announced through text brigade. One stormy day, the principal announces the suspension of classes to two teachers, each of whom sends this message to two other teachers, and so on. Suppose that text messages were sent in five rounds, counting the principal's text message as the first, how many text messages were sent in all?
A. 31 B. 32 C. 63 D. 64

Part II

Read and understand the situation below, then answer the questions or perform the tasks that follow.

Hold on to **HOPE**

Because of the super typhoon Yolanda, there was a big need for blood donors, medicines, doctors, nurses, medical aides, or any form of medical assistance. The Red Cross planned to involve different agencies, organizations, and offices, public and private, local and international, in their project to have massive medical services. The Red Cross contacted first three of the biggest networks, and each of these networks contacted three other networks, and agencies, organizations, and offices, and so on, until enough of these were contacted. It took one hour for an organization to contact three other organizations and all the contacts made were completed within 4 hours. Assume that no group was contacted twice.

1. Suppose you are one of the people in the Red Cross who visualized this project. How many organizations do you think were contacted in the last round? How many organizations were contacted within 4 hours?

2. Make a table to represent the number of organizations, agencies, and offices who could have been contacted in each round.

3. Write an equation to represent the situation. Let the independent variable be the number of rounds and the dependent variable be the number of organizations, agencies, and offices that were contacted in that round.

4. If another hour was used to contact more organizations, how many additional organizations, agencies, and offices could be contacted?

5. Use the given information in the above situation to formulate problems involving these concepts.

6. Write the necessary equations that describe the situations or problems that you formulated.

7. Solve the problems that you formulated.

Rubric for the Equations Formulated and Solved

Score	Descriptors
4	Equations are properly formulated and solved correctly.
3	Equations are properly formulated but not all are solved correctly.
2	Equations are properly formulated but are not solved correctly.
1	Equations are properly formulated but are not solved at all.

Rubric for the Problems Formulated and Solved

Score	Descriptors
6	Poses a more complex problem with two or more solutions and communicates ideas unmistakably, shows in-depth comprehension of the pertinent concepts and/or processes and provides explanation wherever appropriate
5	Poses a more complex problem and finishes all significant parts of the solution and communicates ideas unmistakably, shows in-depth comprehension of the pertinent concepts and/or processes
4	Poses a complex problem and finishes all significant parts of the solution and communicates ideas unmistakably, shows in-depth comprehension of the pertinent concepts and/or processes
3	Poses a complex problem and finishes most significant parts of the solution and communicates ideas unmistakably, shows comprehension of major concepts although neglects or misinterprets less significant ideas or details
2	Poses a problem and finishes some significant parts of the solution and communicates ideas unmistakably but shows gaps on the theoretical comprehension
1	Poses a problem but demonstrates minor comprehension, not being able to develop an approach

Source: D.O. #73, s. 2012

IV. LEARNING GOALS AND TARGETS

After using this module, you should be able to demonstrate understanding of sequences like arithmetic sequences, geometric sequences, and other types of sequences and solve problems involving sequences.

Lesson 1 Arithmetic Sequences

What to Know

In this lesson, you will work with patterns. Recognizing and extending patterns are important skills needed for learning concepts related to an arithmetic sequence.

Activity 1: What's next?

Each item below shows a pattern. Answer the given questions.

1. What is the next shape?

   □, △, □, △, △, □, △, △, △, □, △, △, △, △

   ___

2. What is the next number?
   What is the 10th number?

   0, 4, 8, 12, 16, ____

3. What is the next number?
   What is the 8th number?

   9, 4, -1, -6, -11, ____

4. What is the next number?
   What is the 12th number?

   1, 3, 9, 27, 81, _____

5. What is the next number?
   What is the 7th number?

   160, 80, 40, 20, 10, _____

The set of shapes and the sets of numbers in the above activity are called sequences.

Were you able to find patterns and get the next number in the sequence? Let us now give the formal definition of a sequence.

What is a sequence?

A **sequence** is a function whose domain is the finite set {1, 2, 3,…, *n*} or the infinite set {1, 2, 3,… }.

$$\text{Example:}\quad \begin{array}{cccccc} n & 1 & 2 & 3 & 4 & 5 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a(n) & 3 & -1 & 1.5 & 10 & \pi \end{array}$$

This finite sequence has 5 terms. We may use the notation $a_1,\ a_2,\ a_3,\ ...,\ a_n$ to denote $a(1),\ a(2),\ a(3),\ ...,\ a(n)$, respectively.

In Grade 10, we often encounter sequences that form a pattern such as that found in the sequence below.

$$\text{Example:}\quad \begin{array}{cccccc} n & 1 & 2 & 3 & 4 & ... \\ & \downarrow & \downarrow & \downarrow & \downarrow & \\ a_n & 4 & 7 & 10 & 13 & ... \end{array}$$

The above sequence is an infinite sequence where $a_n = 3n + 1$

In the next two activities, you will learn more about sequences. A general term or *n*th term will be given to you and you will be asked to give the next few terms. You will also be asked to give the *n*th term or the rule for a particular sequence. You may now start with Activity 2.

Activity 2: Term after Term

Find the first 5 terms of the sequence given the *n*th term.

1. $a_n = n + 4$
2. $a_n = 2n - 1$
3. $a_n = 12 - 3n$
4. $a_n = 3^n$
5. $a_n = (-2)^n$

How did you find the activity? Did you find it easy to give the first 5 terms of each sequence? In Activity 3, you will be given the terms of a sequence and you will be asked to find its *n*th term. You may now do Activity 3.

Activity 3: Getting to Know You

What is the *n*th term for each sequence below?

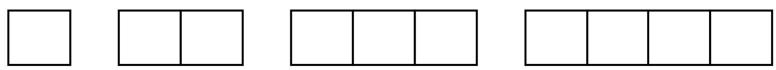
1. 3, 4, 5, 6, 7, ...
2. 3, 5, 7, 9, 11, ...
3. 2, 4, 8, 16, 32, ...
4. -1, 1, -1, 1, -1, ...
5. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

In the activities you have just done, you were able to enumerate the terms of a sequence given its *n*th term and vice versa. Knowing all these will enable you to easily understand a particular sequence. This sequence will be discussed after doing the following activity.

Activity 4: What do we have in common?

We need matchsticks for this group activity. Form a group of 3 students.

1. Below are squares formed by matchsticks.



2. Count the number of matchsticks in each figure and record the results in a table.

number of squares	1	2	3	4	5	6	7	8	9	10
number of matchsticks										

Guide Questions

1. Is there a pattern in the number of matchsticks? If there is, describe it.
2. How is each term (number of matchsticks) found?
3. What is the difference between any two consecutive terms?

How was the activity? What new thing did you learn from the activity?

The above activity illustrates a sequence where the difference between any two consecutive terms is a constant. This constant is called the **common difference** and the said sequence is called an **arithmetic sequence**.

An **arithmetic sequence** is a sequence where every term after the first is obtained by adding a constant called the common difference.

The sequences 1, 4, 7, 10, ... and 15, 11, 7, 3, ... are examples of arithmetic sequences since each one has a common difference of 3 and -4, respectively.

Is the meaning of arithmetic sequence clear to you? Are you ready to learn more about arithmetic sequences? If so, then you have to perform the next activity.

Activity 5: More of the Matchstick Activity

Let us go back to Activity 4. With your groupmates, take a look at the completed table below.

number of squares	1	2	3	4	5	6	7	8	9	10
number of matchsticks	4	7	10	13	16	19	22	25	28	31

Let us take the number of matchsticks 4, 7, 10, 13, 16, 19, 22, 25, 28, and 31. We see that the number of matchsticks forms an arithmetic sequence. Suppose we want to find the 20th, 50th, and 100th terms of the sequence. How do we get them? Do you think a formula would help? If so, we could find a formula for the *n*th term of the sequence. In this case, it will not be difficult since we know the common difference of the sequence.

Let us take the first four terms. Let $a_1 = 4,\ a_2 = 7,\ a_3 = 10,\ a_4 = 13.$ How do we obtain the second, third, and fourth terms?

Consider the table below and complete it. Observe how each term is rewritten.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	...	a_n
4	4+3	4+3+3	4+3+3+3					...	

How else can we write the terms? Study the next table and complete it.

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	...	a_n
4+0(3)	4+1(3)	4+2(3)	4+3(3)					...	

What is a_5? a_{20}? a_{50}?

What is the formula for determining the number of matchsticks needed to form *n* squares?

In general, the first *n* terms of an arithmetic sequence with a_1 as first term and *d* as common difference are

$$a_1,\ a_1 + d,\ a_1 + 2d,\ ...,\ a_1 + (n-1)d.$$

If a_1 and d are known, it is easy to find any term in an arithmetic sequence by using the rule

$$a_n = a_1 + (n-1)d.$$

Example: What is the 10th term of the arithmetic sequence 5, 12, 19, 26, ...?

Solution: Since $a_1 = 5$ and $d = 7$, then $a_{10} = 5 + (10-1)(7) = 68$.

How did you find the activity? The rule for finding the *n*th term of an arithmetic sequence is very useful in solving problems involving arithmetic sequence.

Activity 6: What is missing?

A. Find the missing terms in each arithmetic sequence.

1. 3, 12, 21, __, __, __
2. 8, 3, −2, __, __
3. 5, 12, __, 26, __
4. 2, __, 20, 29, __
5. __, 4, 10, 16, __
6. 17, 14, __, __, 5
7. 4, __, __, 19, 24, ...
8. __, __, __, 8, 12, 16
9. −1, __, __, __, 31, 39
10. 13, __, __, __, −11, −17

B. Find three terms between 2 and 34 of an arithmetic sequence.

Were you able to get the missing terms in each sequence in Part A? Were you able to get the 3 terms in Part B? Let us discuss a systematic way of finding missing terms of an arithmetic sequence.

Finding a certain number of terms between two given terms of an arithmetic sequence is a common task in studying arithmetic sequences. The terms between any two nonconsecutive terms of an arithmetic sequence are known as **arithmetic means**.

Example: Insert 4 arithmetic means between 5 and 25.

Solution: Since we are required to insert 4 terms, then there will be 6 terms in all.

Let $a_1 = 5$ and $a_6 = 25$. We will insert a_2, a_3, a_4, a_5 as shown below:

$$5, \ \underline{a_2}\ , \ \underline{a_3}\ , \ \underline{a_4}\ , \ \underline{a_5}\ , \ 25$$

We need to get the common difference. Let us use $a_6 = a_1 + 5d$ to solve for *d*. Substituting the given values for a_6 and a_1, we obtain $25 = 5 + 5d$. So, $d = 4$.

Using the value of *d*, we can now get the values of $a_2, a_3, a_4,$ and a_5. Thus, $a_2 = 5 + 4(1) = 9$, $a_3 = 5 + 4(2) = 13$, $a_4 = 5 + 4(3) = 17$, and $a_5 = 5 + 4(4) = 21$.

The 4 arithmetic means between 5 and 25 are 9, 13, 17, and 21.

> At this point, you know already some essential things about arithmetic sequence. Now, we will learn how to find the sum of the first *n* terms of an arithmetic sequence. Do Activity 7.

Activity 7: Summing Up

What is the sum of the terms of each finite sequence below?

1. 1, 4, 7, 10
2. 3, 5, 7, 9, 11
3. 10, 5, 0, -5, -10, -15
4. 81, 64, 47, 30, 13, -4
5. -2, -5, -8, -11, -14, -17

Activity 8: The Secret of Karl

What is 1 + 2 + 3 + ... + 50 + 51 + ... + 98 + 99 + 100?

A famous story tells that this was the problem given by an elementary school teacher to a famous mathematician to keep him busy. Do you know that he was able to get the sum within seconds only? Can you beat that? His name was Karl Friedrich Gauss (1777-1885). Do you know how he did it? Let us find out by doing the activity below.

Think-Pair-Share

Determine the answer to the above problem. Then look for a partner and compare your answer with his/her answer. Discuss with him/her your technique (if any) in getting the answer quickly. Then with your partner, answer the questions below and see if this is similar to your technique.

1. What is the sum of each of the pairs 1 and 100, 2 and 99, 3 and 98, ..., 50 and 51?
2. How many pairs are there in #1?
3. From your answers in #1 and #2, how do you get the sum of the integers from 1 to 100?
4. What is the sum of the integers from 1 to 100?

Let us now denote the sum of the first *n* terms of an arithmetic sequence $a_1 + a_2 + a_3 + ... + a_n$ by S_n.

We can rewrite the sum in reverse order, that is,

$$S_n = a_n + a_{n-1} + a_{n-2} + ... + a_1.$$

Rewriting the two equations above using their preceding terms and the difference *d*, we would have

Equation 1: $S_n = a_1 + (a_1 + d) + (a_1 + 2d) + ... + [a_1 + (n-1)d]$

Equation 2: $S_n = a_n + (a_n - d) + (a_n - 2d) + ... + [a_n - (n-1)d]$

Adding equation 1 and equation 2, we get

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + ... + (a_1 + a_n).$$

Since there are *n* terms of the form $a_1 + a_n$, then $2S_n = n(a_1 + a_n)$.

Dividing both sides by 2, we have $S_n = \frac{n}{2}(a_1 + a_n)$.

Now, since we also know that $a_n = a_1 + (n-1)d$, then by substitution, we have

$$S_n = \frac{n\left[a_1 + \left(a_1 + (n-1)d\right)\right]}{2} \text{ or } S_n = \frac{n}{2}[2a_1 + (n-1)d].$$

Example 1: Find the sum of the first 10 terms of the arithmetic sequence 5, 9, 13, 17, ...

Solution: $S_{10} = \frac{10}{2}\left[2(5) + (10-1)4\right] = 230$

Example 2: Find the sum of the first 20 terms of the arithmetic sequence $-2, \ -5, \ -8, \ -11, \ ...$

Solution: $S_{20} = \frac{20}{2}\left[2(-2) + (20-1)(-3)\right] = -610$

How did you find Activity 7? Did you learn many things about arithmetic sequences?

Web Links Learn more about arithmetic sequences through the web. You may open the following links:	http://coolmath.com/algebra/19-sequences-series/05-arithmetic-sequences-01.html http://www.mathisfun.com/algebra/sequences-series.html http://www.mathguide.com/lessons/SequenceArithmetic.html#identify

What to Process

Your goal in this section is to apply the key concepts of arithmetic sequence. Use the mathematical ideas and the examples presented in the preceding section to answer the activities provided.

Activity 9: How well do you know me?

Which of the following sequences is an arithmetic sequence? Why?

1. 3, 7, 11, 15, 19
2. 4, 16, 64, 256
3. 48, 24, 12, 6, 3, ...
4. 1, 4, 9, 16, 25, 36
5. $1, \frac{1}{2}, 0, -\frac{1}{2}$
6. $-2, 4, -8, 16, \ldots$
7. $1, 0, -1, -2,, -3$
8. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
9. $3x, x, \frac{x}{3}, \frac{x}{9}, \ldots$
10. 9.5, 7.5, 5.5, 3.5, ...

Did you find it easy to determine whether a sequence is arithmetic or not? Were you able to give a reason why?

The next activity will assess your skill in using the *n*th term of an arithmetic sequence. You may start the activity now.

Activity 10: More on Arithmetic Sequences

Use the *n*th term of an arithmetic sequence $a_n = a_1 + (n-1)d$ to answer the following questions.

1. Find the 25th term of the arithmetic sequence 3, 7, 11, 15, 19,...
2. The second term of an arithmetic sequence is 24 and the fifth term is 3. Find the first term and the common difference.

3. Give the arithmetic sequence of 5 terms if the first term is 8 and the last term is 100.
4. Find the 9th term of the arithmetic sequence with $a_1 = 10$ and $d = -\frac{1}{2}$.
5. Find a_1 if $a_8 = 54$ and $a_9 = 60$.
6. How many terms are there in an arithmetic sequence with a common difference of 4 and with first and last terms 3 and 59, respectively?
7. Which term of the arithmetic sequence is –18, given that $a_1 = 7$ and $a_2 = 2$?
8. How many terms are in an arithmetic sequence whose first term is -3, common difference is 2, and last term is 23?
9. What must be the value of k so that $5k-3$, $k+2$, and $3k-11$ will form an arithmetic sequence?
10. Find the common difference of the arithmetic sequence with $a_4 = 10$ and $a_{11} = 45$.

Did you find the activity challenging? The next activity is about finding arithmetic means. Remember the *n*th term of an arithmetic sequence.

You may now do Activity 11.

Activity 11: **What can you insert?**

A. Insert the indicated number of arithmetic means between the given first and last terms of an arithmetic sequence.

1.	2 and 32	[1]
2.	6 and 54	[3]
3.	68 and 3	[4]
4.	10 and 40	[5]
5.	$\frac{1}{2}$ and 2	[2]
6.	–4 and 8	[3]
7.	–16 and –8	[3]
8.	$\frac{1}{3}$ and $\frac{11}{3}$	[4]
9.	a and b	[1]
10.	$x+y$ and $4x-2y$	[2]

B. Solve the following problems.

1. The arithmetic mean between two terms in an arithmetic sequence is 39. If one of these terms is 32, find the other term.
2. If five arithmetic means are inserted between -9 and 9, what is the third mean?
3. What are the first and last terms of an arithmetic sequence when its arithmetic means are 35, 15, and -5?
4. Find the value of *x* if the arithmetic mean of 3 and $3x+5$ is 8.
5. Find the value of *a* when the arithmetic mean of $a+7$ and $a+3$ is $3a+9$.

> Did you find the *n*th term of an arithmetic sequence helpful in finding the arithmetic means?
>
> The next activity is about finding the sum of the terms of an arithmetic sequence. You may now proceed.

Activity 12: SUMthing to Do

A. Find the sum of each of the following.

1. integers from 1 to 50
2. odd integers from 1 to 100
3. even integers between 1 and 101
4. first 25 terms of the arithmetic sequence 4, 9, 14, 19, 24, ...
5. multiples of 3 from 15 to 45
6. numbers between 1 and 81 which are divisible by 4
7. first 20 terms of the arithmetic sequence –16, –20, –24, …
8. first 10 terms of the arithmetic sequence 10.2, 12.7, 15.2, 17.7, …
9. 1 + 5 + 9 + … + 49 + 53
10. $\frac{1}{2}+\frac{3}{2}+\frac{5}{2}+\ldots+\frac{17}{2}+\frac{19}{2}$

B. The sum of the first 10 terms of an arithmetic sequence is 530. What is the first term if the last term is 80? What is the common difference?

C. The third term of an arithmetic sequence is –12 and the seventh term is 8. What is the sum of the first 10 terms?

D. Find the sum of the first 25 multiples of 8.

E. Find the sum of the first 12 terms of the arithmetic sequence whose general term is $a_n = 3n + 5$.

Were you able to answer Activity 12? In this section, you were provided with activities to assess your knowledge and skill in what you learned in the previous section. Now that you know the important ideas about arithmetic sequences, let us go deeper by moving to the next section.

What to Reflect on and Understand

Activity 13: **Beyond the Basics**

Do each of the following.

1. Mathematically speaking, the next term cannot be determined by giving only the first finite number of terms of a general sequence. Explain this fact by giving an example.

2. Make a concept map for arithmetic sequences.

3. Using the formula for arithmetic sequence, $a_n = a_1 + (n-1)d$, give problems where the unknown value is (a) a_1, (b) a_n, (c) d and show how each can be found.

4. What should be the value of x so that x + 2, 3x – 2, 7x – 12 will form an arithmetic sequence? Justify your answer.

5. Find the value of x when the arithmetic mean of x + 2 and 4x + 5 is 3x + 2.

6. It is alarming that many people now are being infected by HIV. As the president of the student body in your school, you invited people to give a five-day series of talks on HIV and its prevention every first Friday of the month from 12 noon to 1 p.m. in the auditorium. On the first day, 20 students came. Finding the talk interesting, these 20 students shared the talk to other students and 10 more students came on the second day, another 10 more students came on the third day, and so on.

    a. Assuming that the number of participants continues to increase in the same manner, make a table representing the number of participants from day 1 of the talk until day 5.

    b. Represent the data in the table using a formula. Use the formula to justify your data in the table.

    c. You feel that there is still a need to extend the series of talks, so you decided to continue it for three more days. If the pattern continues where there are 10 additional students for each talk, how many students in all attended the talk on HIV?

> Were you able to accomplish the activity? How did you find it?
>
> You may further assess your knowledge and skill by trying another activity.

Try This:

After a knee surgery, your trainer tells you to return to your jogging program slowly. He suggests jogging for 12 minutes each day for the first week. Each week, thereafter, he suggests that you increase that time by 6 minutes per day. On what week will it be before you are up to jogging 60 minutes per day?

> Were you able to solve the problem?
>
> Now that you have a deeper understanding of the topic, you are now ready to do the tasks in the next section.

What to Transfer

Your goal in this section is to apply what you learned to real-life situations. You will be given a task which will demonstrate your understanding of arithmetic sequences.

Activity 14: **Reality Series**

In groups of five, create a well-developed Reality Series considering the following steps:

1. Choose a real-life situation which involves arithmetic sequences. You could research online or create your own. Be sure to choose what interests your group the most to make your Reality Series not only interesting but also entertaining.
2. Produce diagrams or pictures that will help others see what is taking place in the situation or the scenario that you have chosen.
3. Prepare the necessary table to present the important data in your situation and the correct formula and steps to solve the problem.
4. Show what you know about the topic by using concepts about arithmetic sequences to describe the situation. For example, show how to find the *n*th term of your arithmetic sequence or find the sum of the first *n* terms. Write your own questions about the situation and be ready with the corresponding answers.
5. Present your own Reality Series in the class.

How did the task help you realize the importance of the topic in real life?

Rubric for the Written Report about Chosen Real-Life Situation

Score	Descriptors
5	The written report is completely accurate and logically presented/designed. It includes facts, concepts, and computations involving arithmetic sequences. The chosen real-life situation is very timely and interesting.
4	The written report is generally accurate and the presentation/design reflects understanding of arithmetic sequences. Minor inaccuracies do not affect the overall results. The chosen real-life situation is timely and interesting.
3	The written report is generally accurate but the presentation/design lacks application of arithmetic sequences. The chosen real-life situation is somehow timely and interesting.
2	The written report contains major inaccuracies and significant errors in some parts. The chosen real-life situation is not timely and interesting.
1	There is no written report made.

Rubric for the Oral Presentation

Score	Descriptors
5	Oral presentation is exceptionally clear, thorough, fully supported with concepts and principles of arithmetic sequences, and easy to follow.
4	Oral report is generally clear and reflective of students' personalized ideas, and some accounts are supported by mathematical principles and concepts of arithmetic sequences.
3	Oral report is reflective of something learned; it lacks clarity and accounts have limited support.
2	Oral report is unclear and impossible to follow, is superficial, and more descriptive than analytical.
1	No oral report was presented.

SUMMARY/SYNTHESIS/GENERALIZATION

This lesson is about arithmetic sequences and how they are illustrated in real life. You learned to:

- generate patterns;
- determine the *n*th term of a sequence;
- describe an arithmetic sequence, and find its *n*th term;
- determine the arithmetic means of an arithmetic sequence;
- find the sum of the first *n* terms of an arithmetic sequence; and
- solve real-life problems involving arithmetic sequence.

Lesson 2 Geometric and Other Sequences

What to Know

The previous lesson focused on arithmetic sequences. In this lesson, you will also learn about geometric sequences and the process on how they are generated. You will also learn about other types of sequences.

Activity 1: Divide and Conquer

Find the ratio of the second number to the first number.

1.	2,	8
2.	–3,	9
3.	1,	$\frac{1}{2}$
4.	–5,	–10
5.	12,	4
6.	–49,	7
7.	$\frac{1}{4}$,	$\frac{1}{2}$
8.	a^2,	a^3
9.	$k-1$,	k
10.	$3m$,	$3mr$

You need the concept of ratio in order to understand the next kind of sequence. We will explore that sequence in the next activity. Do the next activity now.

Activity 2: Fold Me Up

Do the activity with a partner. One of you will perform the paper folding while the other will do the recording in the table.

1. Start with a big square from a piece of paper. Assume that the area of the square is 64 square units.
2. Fold the four corners to the center of the square and find the area of the resulting square.
3. Repeat the process three times and record the results in the table below.

Square	1	2	3
Area			

Guide Questions

1. What is the area of the square formed after the first fold? Second fold? Third fold?
2. Is there a pattern in the areas obtained after 3 folds?
3. You have generated a sequence of areas. What are the first 3 terms of the sequence?
4. Is the sequence an arithmetic sequence? Why?
5. Using the pattern in the areas, what would be the 6th term of the sequence?

The sequence 32, 16, 8, 4, 2, 1 is called a geometric sequence.

A **geometric sequence** is a sequence where each term after the first is obtained by multiplying the preceding term by a nonzero constant called the **common ratio**.

The common ratio, *r*, can be determined by dividing any term in the sequence by the term that precedes it. Thus, in the geometric sequence

32, 16, 8, 4, 2, ... , the common ratio is $\frac{1}{2}$ since $\frac{16}{32} = \frac{1}{2}$.

The next activity will test whether you can identify geometric sequences or not.

Activity 3: I'll Tell You What You Are

State whether each of the following sequences is geometric or not.

1. 5, 20, 80, 320, ...
2. $7\sqrt{2},\ 5\sqrt{2},\ 3\sqrt{2},\ \sqrt{2},...$
3. 5, –10, 20, –40
4. 1, 0.6, 0.36, 0.216, ...
5. $\frac{10}{3}, \frac{10}{6}, \frac{10}{9}, \frac{10}{15}$
6. 4, 0, 0, 0, 0…

Activity 4: The Rule of a Geometric Sequence

Form a group of 3 members and answer the guide questions using the table.

Problem: What are the first 5 terms of a geometric sequence whose first term is 2 and whose common ratio is 3?

Term	Other Ways to Write the Terms	
	In Factored Form	In Exponential Form
$a_1 = 2$	2	2×3^0
$a_2 = 6$	2 x 3	2×3^1
$a_3 = 18$	2 x 3 x 3	2×3^2
$a_4 = 54$	2 x 3 x 3 x 3	2×3^3
$a_5 = 162$	2 x 3 x 3 x 3 x 3	2×3^4
⋮	⋮	⋮
a_n		?

Guide Questions

1. Look at the two ways of writing the terms. What does 2 represent?
2. For any two consecutive terms, what does 3 represent?
3. What is the relationship between the exponent of 3 and the position of the term?
4. If the position of the term is *n*, what must be the exponent of 3?
5. What is a_n for this sequence?
6. In general, if the first term of a geometric sequence is a_1 and the common ratio is *r*, what is the *n*th term of the sequence?

What did you learn in the activity? Given the first term a_1 and the common ratio r of a geometric sequence, the nth term of a geometric sequence is $a_n = a_1 r^{n-1}$.

Example: What is the 10th term of the geometric sequence 8, 4, 2, 1, ...?

Solution: Since $r = \frac{1}{2}$, then $a_{10} = 8\left(\frac{1}{2}\right)^9 = 8\left(\frac{1}{512}\right) = \frac{1}{64}$.

> In the next activity, you will find the nth term of a geometric sequence, a skill that is useful in solving other problems involving geometric sequences. Do the next activity.

Activity 5: Missing You

A. Find the missing terms in each geometric sequence.

1. 3, 12, 48, ___, ___
2. ___, ___, 32, 64, 128, ...
3. 120, 60, 30, ___, ___, ___
4. 5, ___, 20, 40, ___, ___
5. ___, 4, 12, 36, ___, ___
6. –2, ___, ___, –16 –32 –64
7. 256, ___, ___, –32 16, ...
8. 27, 9, ___, ___, $\frac{1}{3}$
9. $\frac{1}{4}$, ___, ___, ___, 64, 256
10. $5x^2$ ___, $5x^6$ $5x^8$ ___ , ...

B. Insert 3 terms between 2 and 32 of a geometric sequence.

Were you able to answer the activity? Which item in the activity did you find challenging? Let us now discuss how to find the geometric means between terms of a geometric sequence.

Inserting a certain number of terms between two given terms of a geometric sequence is an interesting activity in studying geometric sequences. We call the terms between any two given terms of a geometric sequence the **geometric means**.

Example: Insert 3 geometric means between 5 and 3125.

Solution:

Let $a_1 = 5$ and $a_5 = 3125$. We will insert a_2, a_3, and a_4.

Since $a_5 = a_1 r^4$, then $3125 = 5r^4$.

Solving for the value of *r*, we get $625 = r^4$ or $r = \pm 5$.

We obtained two values of *r*, so we have two geometric sequences.

If $r = 5$, the geometric means are
$a_2 = 5(5)^1 = 25$, $a_3 = 5(5)^2 = 125$, $a_4 = 5(5)^3 = 625$.

Thus, the sequence is 5, 25, 125, 625, 3125.

If $r = -5$, then the geometric means are
$a_2 = 5(-5)^1 = -25$, $a_3 = 5(-5)^2 = 125$, $a_4 = 5(-5)^3 = -625$.

Thus, the sequence is $5, -25, 125, -625, 3125$.

At this point, you already know some essential ideas about geometric sequences. Now, we will learn how to find the sum of the first *n* terms of a geometric sequence. Do Activity 6.

Activity 6: Want Sum?

Do the following with a partner.

Part 1:

Consider the geometric sequence 3, 6, 12, 24, 48, 96, ...
What is the sum of the first 5 terms?

There is another method to get the sum of the first 5 terms.

Let $S_5 = 3 + 6 + 12 + 24 + 48$.

Multiplying both sides by the common ratio 2, we get
$2S_5 = 6 + 12 + 24 + 48 + 96$

Subtracting $2S_5$ from S_5, we have

$$\begin{array}{rl} S_5 = & 3 + 6 + 12 + 24 + 48 \\ -\ (2S_5 = & \quad\ \ 6 + 12 + 24 + 48 + 96) \\ \hline -S_5 = & 3 \qquad\qquad\qquad\qquad\quad -96 \end{array}$$

$$-S_5 = -93$$
$$S_5 = 93$$

Try the method for the sequence 81, 27, 9, 3, 1, ... and find the sum of the first 4 terms.

From the activity, we can derive a formula for the sum of the first *n* terms, S_n, of a geometric sequence.

Consider the sum of the first n terms of a geometric sequence:
$S_n = a_1 + a_1r + a_1r^2 + \dots + a_1r^{n-1}$ (equation 1)

Multiplying both sides of equation 1 by the common ratio r, we get
$rS_n = a_1r + a_1r^2 + a_1r^3 + \dots + a_1r^{n-1} + a_1r^n$ (equation 2)

Subtracting equation 2 from equation 1, we get

$$\begin{array}{rlr} S_n = & a_1 + a_1r + a_1r^2 + \dots + a_1r^{n-1} & \text{equation 1} \\ -\left(rS_n = \right. & \left. a_1r + a_1r^2 + \dots + a_1r^{n-1} + a_1r^n\right) & \text{equation 2} \\ \hline S_n - rS_n = & a_1 \qquad\qquad\qquad - a_1r^n & \end{array}$$

Factoring both sides of the resulting equation, we get
$$S_n(1-r) = a_1(1-r^n).$$

Dividing both sides by $1-r$, where $1-r \neq 0$, we get
$$S_n = \frac{a_1(1-r^n)}{1-r}, \; r \neq 1.$$

Note that since $a_n = a_1r^{n-1}$, if we multiply both sides by r we get
$$a_n(r) = a_1r^{n-1}(r) \text{ or } a_nr = a_1r^n.$$

Since $S_n = \dfrac{a_1(1-r^n)}{1-r} = \dfrac{a_1 - a_1r^n}{1-r}$,

Then replacing a_1r^n by a_nr, we have
$$S_n = \frac{a_1 - a_nr}{1-r}, \; r \neq 1.$$

What if $r = 1$?

If $r = 1$, then the formula above is not applicable. Instead,
$$S_n = a_1 + a_1(1) + a_1(1)^2 + \dots + a_1(1)^{n-1} = \underbrace{a_1 + a_1 + a_1 + \dots + a_1}_{n \text{ terms}} = na_1.$$

Example: What is the sum of the first 10 terms of $2+2+2+...$?

Solution: $2+2+2+2+2+2+2+2+2+2=10(2)=20$

What if $r=-1$?

If $r=-1$ and n is even, then

$$\begin{aligned} S_n &= a_1 + a_1(-1) + a_1(-1)^2 + a_1(-1)^3 + ... + a_1(-1)^{n-1} \\ &= a_1 - a_1 + a_1 - a_1 + ... + a_1 - a_1 \\ &= (a_1 - a_1) + (a_1 - a_1) + ... + (a_1 - a_1) \\ &= 0 \end{aligned}$$

However, if $r=-1$ and n is odd, then

$$\begin{aligned} S_n &= a_1 + a_1(-1) + a_1(-1)^2 + a_1(-1)^3 + ... + a_1(-1)^{n-1} \\ &= a_1 - a_1 + a_1 - a_1 + ... + a_1 - a_1 + a_1 \\ &= (a_1 - a_1) + (a_1 - a_1) + ... + (a_1 - a_1) + a_1 \\ &= a_1 \end{aligned}$$

To summarize, $S_n = \begin{cases} \dfrac{a_1(1-r^n)}{1-r} \text{ or } \dfrac{a_1 - a_n r}{1-r}, & \text{if } r \neq 1 \\ na_1, & \text{if } r = 1 \end{cases}$

In particular, if $r=-1$ the sum S_n simplifies to

$$S_n = \begin{cases} 0 \text{ if } n \text{ is even} \\ a_1 \text{ if } n \text{ is odd} \end{cases}$$

Example 1: What is the sum of the first 10 terms of $2-2+2-2+...$?
Solution: Since $r=-1$ and n is even, then the sum is 0.

Example 2: What is the sum of the first 11 terms of $2-2+2-2+...$?
Solution: Since $r=-1$ and n is odd, then the sum is 2.

Example 3: What is the sum of the first five terms of 3, 6, 12, 24, 48, 96,...?

Solution: Since $a_1 = 3,\ r = 2,$ and $n = 5,$ then the sum is

$$S_5 = \frac{3\left(1-2^5\right)}{1-2} = \frac{3(-31)}{-1} = 93.$$

Alternative Solution: Using $S_n = \dfrac{a_1 - a_n r}{1-r},$ let $a_1 = 3,\ a_5 = 48,$ and $r = 2.$ Then

$$S_5 = \frac{3-(48)(2)}{1-2} = \frac{3-96}{-1} = \frac{-93}{-1} = 93.$$

Part 2:

Is it possible to get the sum of an infinite number of terms in a geometric sequence?

Consider the infinite geometric sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

If we use the formula $S_n = \dfrac{a_1\left(1-r^n\right)}{1-r},$ then

$$S_n = \frac{\frac{1}{2}\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} = \frac{\frac{1}{2}-\frac{1}{2}\left(\frac{1}{2}\right)^n}{\frac{1}{2}} = 2\left[\frac{1}{2}-\frac{1}{2}\left(\frac{1}{2}\right)^n\right] = 1-\left(\frac{1}{2}\right)^n.$$

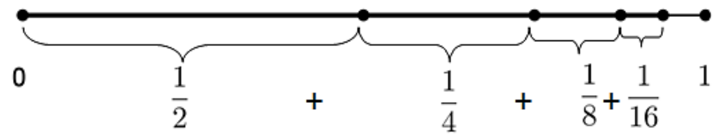
The first five values of S_n are shown in the table below.

n	1	2	3	4	5
S_n	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	$\frac{31}{32}$

What happens to the value of S_n as *n* gets larger and larger?

Observe that S_n approaches 1 as *n* increases, and we say that *S* = 1.

To illustrate further that the sum of the given sequence is 1, let us show the sum of the sequence $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$ on a number line, adding one term at a time:



What does this tell us? Clearly, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=1$.

We call the sum that we got as the **sum to infinity**. Note that the common ratio in the sequence is $\frac{1}{2}$, which is between –1 and 1. We will now derive the formula for the sum to infinity when $-1<r<1$.

Recall that $S_n=\frac{a_1\left(1-r^n\right)}{1-r}=\frac{a_1}{1-r}-\frac{a_1r^n}{1-r}$. Suppose that $-1<r<1$. As the number of terms becomes larger, that is, as n approaches infinity, then r^n approaches 0, and $\frac{a_1r^n}{1-r}$ approaches 0. Thus, the sum of the terms of an infinite geometric sequence $a_1, a_1r, a_1r^2,\ldots,$ where $-1<r<1$ is given by the formula

$$S=\frac{a_1}{1-r}.$$

This formula is also known as the **sum to infinity**.

Example 1: What is the sum to infinity of $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16},\ldots$?

Solution: Since $a_1=\frac{1}{2}$ and $r=\frac{1}{2}$, then $S=\dfrac{\frac{1}{2}}{1-\frac{1}{2}}=1$.

Example 2: What is the sum to infinity of $3-1+\frac{1}{3}-\frac{1}{9}+\ldots$?

Solution: Since $a_1=3$ and $r=-\frac{1}{3}$, then $S=\dfrac{3}{1-\left(-\frac{1}{3}\right)}=\frac{9}{4}$.

You have already learned how to find the sum of the terms of an infinite geometric sequence, where $|r| < 1$, that is, $-1 < r < 1$. What if $|r| \geq 1$, that is, $r \geq 1$ or $r \leq -1$? Let us find out by performing the next activity.

Part 3:

Consider the infinite geometric sequence 2, 4, 8, 16, 32, 64, ... Complete the table below by finding the indicated partial sums. Answer the questions that follow.

S_1	S_2	S_3	S_4	S_5

Guide Questions

1. What is the common ratio of the given sequence?
2. What happens to the values of S_n as *n* increases?
3. Does the given infinite sequence have a finite sum?

Note that if $r \geq 1$, the values of S_n are not guaranteed to approach a finite number as *n* approaches infinity.

Consider the infinite geometric sequence 5, –25, 125, –625,… Complete the table below by finding the indicated partial sums. Answer the questions that follow.

S_1	S_2	S_3	S_4	S_5

Guide Questions

1. What is the common ratio of the given sequence?
2. What happens to the values of S_n as *n* increases?
3. Does the given sequence have a finite sum?

Note that if $r \leq -1$, the values of S_n are not guaranteed to approach a finite number.

The above activities indicate that whenever $|r| \geq 1$, that is, $r \geq 1$ or $r \leq -1$, the sum of the terms of an infinite geometric sequence does not exist.

Did you learn many things about geometric sequences?

Activity 7: **How well do you know me?**

Determine whether each sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, give the common difference; if geometric, give the common ratio.

1. 6, 18, 54, 162, ...
2. 4, 10, 16, 22, ...
3. 1, 1, 2, 3, 5, 8, ...
4. 625, 125, 25, 5, …
5. $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$
6. 5, 8, 13, 21, 34, ...
7. –1296, 216, –36, 6, …
8. 8.2, 8, 7.8, 7.6, ...
9. $-\frac{1}{42}, -\frac{1}{35}, -\frac{1}{28}, -\frac{1}{21}, \ldots$
10. $11, 2, -7, -16, \ldots$

The sequences in numbers 3, 5, 6, and 9 are neither arithmetic nor geometric. The sequences in numbers 5 and 9 which are $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$ and $-\frac{1}{42}, -\frac{1}{35}, -\frac{1}{28}, -\frac{1}{21}, \ldots$, respectively, are called harmonic sequences while the sequences in numbers 3 and 6 which are 1, 1, 2, 3, 5, 8, ... and 5, 8, 13, 21, 34, ... , respectively, are parts of what we call a Fibonacci sequence. These are other types of sequences.

What is a harmonic sequence?

A **harmonic sequence** is a sequence such that the reciprocals of the terms form an arithmetic sequence.

If we take the reciprocals of the terms of the harmonic sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$ then the sequence becomes $2, 4, 6, 8, \ldots$ which is an arithmetic sequence. What is the next term in the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$?

Example 1: Given the arithmetic sequence $-20, -26, -32, -38, \ldots$, find the first 8 terms of the corresponding harmonic sequence.

Solution: Completing the 8 terms of the given sequence, we have $-20, -26, -32, -38, -44, -50, -56, -62$.

Therefore, the first 8 terms of the harmonic sequence are $-\frac{1}{20}, -\frac{1}{26}, -\frac{1}{32}, -\frac{1}{38}, -\frac{1}{44}, -\frac{1}{50}, -\frac{1}{56}, -\frac{1}{62}$.

Example 2: Given the arithmetic sequence $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, find the 10th term of the corresponding harmonic sequence.

Solution: Getting the 10th term of the given sequence which is 5, then the 10th term of the harmonic sequence is $\frac{1}{5}$.

What is a Fibonacci sequence?

A **Fibonacci sequence** is a sequence where its first two terms are either both 1, or 0 and 1; and each term, thereafter, is obtained by adding the two preceding terms.

What is the next term in the Fibonacci sequence $0, 1, 1, 2, 3, 5, \ldots$?

Example: Given the Fibonacci sequence $5, 8, 13, 21, 34, \ldots$, find the next 6 terms.

Solution: Since each new term in a Fibonacci sequence can be obtained by adding its two preceding terms, then the next 6 terms are 55, 89, 144, 233, 377, and 610.

You are now acquainted with four kinds of sequences: arithmetic, geometric, harmonic, and Fibonacci.

Web Links Learn more about geometric, harmonic, and Fibonacci sequences through the web. You may open the following links:	http://coolmath.com/algebra/19-sequences-series/07-geometric-sequences-01.html http://coolmath.com/algebra/19-sequences-series/08-geometric-series-01.html http://www.mathisfun.com/algebra/sequences-series-sums-geometric.html http://www.mathguide.com/lessons/SequenceGeometric.html csexcelgroup.tripod.com www.mathisfun.com/numbers/fibonacci-sequence.html

What to Process

Your goal in this section is to apply the key concepts of geometric sequences. Use the mathematical ideas and the examples presented in the preceding section to answer the activities provided.

Activity 8: Do you remember me?

State whether the given sequence is arithmetic, geometric, harmonic, or part of a Fibonacci. Then, give the next term of the sequence.

1. 8, 16, 24, 32, ...
2. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$
3. 1296, 216, 36, 6, ...
4. 8, 13, 21, 34, 55, ...

5. $\frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \ldots$

6. $\frac{1}{24}, \frac{1}{20}, \frac{1}{16}, \frac{1}{12}, \ldots$

7. $2\sqrt{2}, 5\sqrt{2}, 8\sqrt{2}, 11\sqrt{2}, \ldots$

8. $\frac{6}{11}, \frac{6}{17}, \frac{6}{23}, \frac{6}{29}, \ldots$

9. $6, -18, 54, -162, \ldots$

10. $40, 8, \frac{8}{5}, \frac{8}{25}, \ldots$

> Was it easy for you to determine which sequence is arithmetic, geometric, harmonic, or Fibonacci? Were you able to give the next term? The next activity will assess your skill in using the *n*th term of a geometric sequence. You may start the activity now.

Activity 9: There's More on Geometric Sequences

Use the *n*th term of a geometric sequence $a_n = a_1 r^{n-1}$ to answer the following questions.

1. What is the 5th term of the geometric sequence $\frac{3}{20}, \frac{3}{2}, 15, \ldots$?
2. Find the sixth term of a geometric sequence where the second term is 6 and the common ratio is 2.
3. Find k so that the terms $k-3$, $k+1$, and $4k-2$ form a geometric sequence.
4. In the geometric sequence 6, 12, 24, 48, ..., which term is 768?
5. The second term of a geometric sequence is $\frac{3}{4}$ and its fourth term is 3.

   What is the first term?

> The next activity is about finding the geometric means. Always remember the *n*th term of a geometric sequence.
>
> You may now proceed to Activity 10.

Activity 10: Finding Geometric Means

A. Find the indicated number of geometric means between each pair of numbers.

1. 16 and 81 [3]
2. 256 and 1 [3]
3. –32 and 4 [2]
4. $\frac{1}{3}$ and $\frac{64}{3}$ [1]
5. $2xy$ and $16xy^4$ [2]

B. The geometric mean between the first two terms in a geometric sequence is 32. If the third term is 4, find the first term.

C. Insert a geometric mean between k and $\frac{1}{k}$.

D. If 2 and 3 are two geometric means between m and n, find the values of m and n.

E. Three positive numbers form a geometric sequence. If the geometric mean of the first two numbers is 6 and the geometric mean of the last two numbers is 24, find the three numbers and their common ratio.

> Was knowing the nth term of a geometric sequence helpful in finding geometric means?
>
> The next activity is about finding the sum of the first n terms of a geometric sequence. You may now proceed.

Activity 11: Sum of Terms in a Geometric Sequence

A. For each given geometric sequence, find the sum of the first:

1. 5 terms of 4, 12, 36, 108, ...
2. 6 terms of 3, –6, 12, –24, …
3. 6 terms of –3, 3, –3, 3, …
4. 7 terms of –3, 3, –3, 3, …
5. 8 terms of $\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \ldots$

B. Find the sum to infinity of each geometric sequence, if it exists.

1. 64, 16, 4, 1, ...
2. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$
3. $-4, -1, -\frac{1}{4}, -\frac{1}{16}, \ldots$
4. $24, 4, \frac{2}{3}, \frac{1}{9}, \ldots$
5. $1, \sqrt{2}, 2, 2\sqrt{2}, \ldots$

C. Find the sum of the terms of a geometric sequence where the first term is 4, the last term is 324, and the common ratio is 3.

D. The sum to infinity of a geometric sequence is twice the first term. What is the common ratio?

What to Reflect on and Understand

Activity 12: More Problems in Geometric Sequences

Do the following.

1. Create a concept web for geometric sequences.

2. Compare and contrast arithmetic and geometric sequences using a two-column chart.

3. Given the geometric sequence 1, 2, 4, 8, 16, 32, …., think of a simple real-life situation which represents this sequence (group activity through “Power of Four”).

4. Find the value of x so that $x+2$, $5x+1$, $x+11$ will form a geometric sequence. Justify your answer. Find the sum of the first 10 terms of the given sequence.

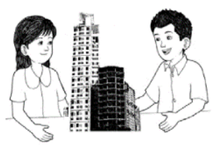
5. Find the value of x if the geometric mean of $2x$ and $19x - 2$ is $7x - 2$.

6. The World Health Organization (WHO) reported that about 16 million adolescent girls between 15 and 19 years of age give birth each year. Knowing the adverse effects of adolescent childbearing on the health of the mothers as well as their infants, a group of students from Magiting High School volunteered to help the government in its campaign for the prevention of early pregnancy by giving lectures to 7 barangays about the WHO Guidelines on teenage pregnancy. The group started in Barangay 1 and 4 girls attended the lecture. Girls from other barangays heard about it, so 8 girls attended from Barangay 2, 16 from Barangay 3, and so on.

   a. Make a table representing the number of adolescent girls who attended the lecture from Barangay 1 to Barangay 7 assuming that the number of attendees doubles at each barangay.

   b. Analyze the data in the table and create a formula. Use the formula to justify your data in the table.

c. Because people who heard about the lecture given by the group thought that it would be beneficial to them, five more different barangays requested the group to do the lectures for them. If the number of young girls who will listen to the lecture from these five barangays will increase in the same manner as that of the first 7 barangays, determine the total number of girls who will benefit from the lecture.



What to Transfer

Activity 13: May the Best Man Win

Do the following by group.

Imagine that you were one of the people in the Human Resource group of a fast growing company in the Philippines. All of you were asked by the management to create a salary scheme for a very important job that the company would offer to the best IT graduates this year. The management gave the salary range good for 2 years, telling everyone in your group that whoever could give a salary scheme that would best benefit both the employer and the would-be employees would be given incentives.

1. Form groups of 5. In your respective groups, make use of all the concepts you learned on geometric sequences considering the starting salary, the rate of increase, the time frame, etc. in making different salary schemes and in deciding which one will be the best for both the employer and the would-be employees.
2. Prepare a visual presentation of your chosen salary scheme with the different data that were used, together with the formulas and all the computations done. You may include one or two salary schemes that you have prepared in your group for comparison.
3. In a simulated board meeting, show your visual presentation to your classmates who will act as the company's human resource administrative officers.

Rubric for the Chosen Salary Scheme and Visual Presentation

Score	Descriptors
5	The salary scheme and visual presentation are completely accurate and logically presented/designed including facts, concepts, and computation involving geometric sequences. The scheme is advantageous to both employer and employees.
4	The salary scheme and visual presentation are generally accurate and the presentation/design reflects understanding of geometric sequences. Minor inaccuracies do not affect the overall results. The scheme is advantageous to both employer and employees.
3	The salary scheme and visual presentation are generally accurate but the presentation/design lacks application of geometric sequences. The scheme is a little bit favorable to the employer.
2	The salary scheme and visual presentation contain major inaccuracies and significant errors in some parts. One cannot figure out which scheme is advantageous.
1	There are no salary scheme and visual presentation made.

Rubric for Presentation

Score	Descriptors
5	Presentation is exceptionally clear, thorough, fully supported with concepts and principles of geometric sequences, and easy to follow.
4	Presentation is generally clear and reflective of students' personalized ideas, and some accounts are supported by mathematical principles and concepts of geometric sequences.
3	Presentation is reflective of something learned; lacks clarity and accounts have limited support.
2	Presentation is unclear and impossible to follow, is superficial, and more descriptive than analytical.
1	No presentation.

Summary/Synthesis/Generalization

This lesson was about geometric sequences and other types of sequences. You learned to:

- distinguish between arithmetic and geometric sequences;
- recognize harmonic and Fibonacci sequences;
- describe a geometric sequence, and find its *n*th term;
- determine the geometric means between two terms;
- find the sum of the terms of a geometric sequence; and
- solve real-life problems involving geometric sequences.

GLOSSARY OF TERMS

Arithmetic Means – terms $m_1, m_2, ..., m_k$ between two numbers a and b such that $a, m_1, m_2, ..., m_k, b$ is an arithmetic sequence

Arithmetic Sequence – a sequence where each term after the first is obtained by adding the same constant

Common Difference – a constant added to each term of an arithmetic sequence to obtain the next term of the sequence

Common Ratio – a constant multiplied to each term of a geometric sequence to obtain the next term of the sequence

Fibonacci Sequence – a sequence where its first two terms are either both 1, or 0 and 1; and each term, thereafter, is obtained by adding the two preceding terms.

Finite Sequence – a function whose domain is the finite set $\{1, 2, 3, ..., n\}$

Geometric Means – terms $m_1, m_2, ..., m_k$ between two numbers a and b such that $a, m_1, m_2, ..., m_k, b$ is a geometric sequence.

Geometric Sequence – a sequence where each term after the first is obtained by multiplying the preceding term by the same constant

Harmonic Sequence – a sequence such that the reciprocals of the terms form an arithmetic sequence

Infinite Sequence – a function whose domain is the infinite set {1, 2, 3, …}

Sequence (of real numbers) – a function whose domain is the finite set {1, 2, 3, …, *n*} or the infinite set {1, 2, 3, …}

Term - any number in a sequence

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http://www.mathguide.com/lessons/SequenceArithmetic.html#identify

http://coolmath.com/algebra/19-sequences-series/07-geometric-sequences-01.html

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2 POLYNOMIALS AND POLYNOMIAL EQUATIONS

I. INTRODUCTION

In Grade 9, you learned about quadratic equations. This module on polynomial equations will extend what you learned in Grade 9.

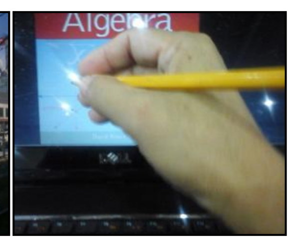










Some real-life situations require the application of polynomials. For example, engineers can use polynomials to create building plans and entrepreneurs can use polynomials to design cost-effective products.

By the end of this module, you are expected to be able to solve polynomial equations. Before that, you need to learn first the basic concepts related to polynomial equations.

II. LESSONS AND COVERAGE

In this module, the following lessons will be discussed:

Lesson 1 – Division of Polynomials

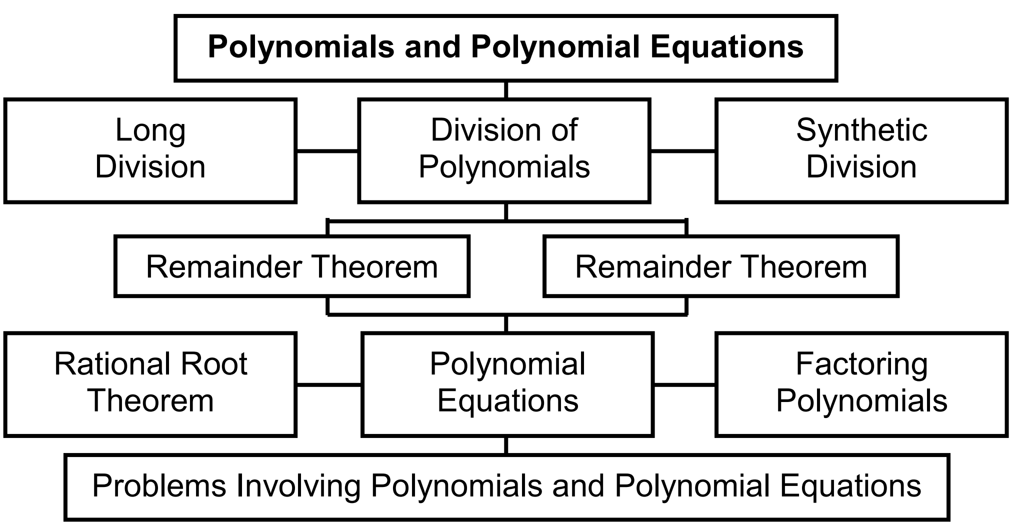
Lesson 2 – The Remainder Theorem and the Factor Theorem

Lesson 3 – Polynomial Equations

In these lessons, you will learn to:

Lesson 1	• perform long division and synthetic division on polynomials
Lesson 2	• prove the Remainder Theorem and the Factor Theorem • factor polynomials
Lesson 3	• illustrate polynomial equations • prove the Rational Roots Theorem • solve polynomial equations • solve problems involving polynomials and polynomial equations

Module Map



III. PRE-ASSESSMENT

Part I

Let us find out first what you already know about the content of this module. Try to answer all items. Take note of the items/questions that you were not able to answer correctly and revisit them as you go through this module for self-correction.

1. Which of the following is a polynomial?

   i. $4x^3 + 9x - 5x^2 + 7$
   ii. $2x^{-5} + x^{-2} + x^{-3} + 2x + 5$
   iii. $\dfrac{1}{x^2 + 3x + 6}$

   A. i only
   B. ii only
   C. i and ii
   D. i and iii

2. The following are examples of polynomials, EXCEPT
   A. $x^2 - 4x + 5$
   B. $4x^{-3} + 8x^{-2} + 10x - 7$
   C. $3x^4 - 5x^3 + 2x - 1$
   D. $x^3 - y^3$

3. The leading coefficient of the polynomial $5x^{10} + 4x^{12} + 4x^6 + x^4 - x$ is
   A. 4
   B. 5
   C. 10
   D. 12

4. What is the quotient when $x^2 - 25$ is divided by $x - 5$?
   A. $x - 5$
   B. $x - 25$
   C. $x + 5$
   D. $x + 25$

   For items 5 to 8, use the illustration on long division that follows:
   Divide $(5x^2 + 14x - 24)$ by $(x + 4)$.

$$
\begin{array}{r|l}
 & \phantom{5x^2+{}}5x - 6 \\
\hline
x + 4 & 5x^2 + 14x - 24 \\
\text{2nd line} & \underline{5x^2 + 20x} \\
 & \phantom{5x^2+{}} -6x - 24 \\
 & \phantom{5x^2+{}} \underline{-6x - 24} \\
 & 0
\end{array}
$$

5. What is the remainder?
   A. $5x - 6$
   B. $x + 4$
   C. -6
   D. 0

6. Which is the divisor?
   A. $x + 4$
   B. $5x - 6$
   C. $5x^2 + 14x - 24$
   D. 0

7. Which is the quotient?
   A. $x + 4$
   B. $5x - 6$
   C. $5x^2 + 14x - 24$
   D. 0

8. What is the process used to obtain the 2nd line?
   A. Subtracting $5x$ from $(x + 4)$
   B. Dividing $5x$ by $(x + 4)$
   C. Adding $5x$ to $(x + 4)$
   D. Multiplying $5x$ by $(x + 4)$

9. Which expression gives the remainder when $P(x) = 4x^2 + 2x - 5$ is divided by $x - 2$?
   A. $P(-5)$
   B. $P(-2)$
   C. $P(2)$
   D. $P\left(\frac{5}{4}\right)$

10. Find the remainder when $(x^9 + 2x^8 + 3x^7 + \ldots + 9x)$ is divided by $(x - 1)$.
    A. 45
    B. 90
    C. 180
    D. 360

11. What is the remainder when $(5x^{100} + 5)$ is divided by $(x - 1)$?
    A. 5
    B.10
    C. -5
    D. -10

12. The remainder after dividing $(-10x^3 + 5x^2 + K)$ by $(x + 1)$ is 4. Which of the following is the value of K?
    A. 9
    B. 19
    C. -19
    D. -11

13. Which of the following polynomials is exactly divisible by $(3x + 1)$?
    A. $6x^2 + 17x + 5$
    B. $9x^2 + 6x + 1$
    C. $3x^3 + 4x^2 - 8x - 3$
    D. all of the above

14. Which of the following is the factored form of $x^3 + 3x^2 - 10x - 24$?

A. $(x + 4)(x - 3)(x + 2)$ C. $(x - 4)(x - 3)(x + 2)$

B. $(x - 4)(x - 3)(x - 2)$ D. $(x + 4)(x + 3)(x - 2)$

15. Which polynomial gives a quotient of $(3x^2 + 2x + 4)$ and a remainder of 19 when divided by $(2x - 3)$?

A. $6x^3 - 5x^2 + 2x$ C. $6x^3 - 5x^2 + 2x + 7$

B. $6x^3 - 5x^2 + 4x + 7$ D. $6x^3 + 5x^2 + 2x + 7$

16. What is the quotient when $(2x^4 + 4x^3 - 5x^2 + 2x - 3)$ is divided by $(2x^2 + 1)$?

A. $x^2 + 2x - 3$ C. $x^2 - 2x - 3$

B. $x^2 - 2x + 3$ D. $x^2 + 2x + 3$

17. Find the value of k so that $(x + 2)$ is a factor of $3x^3 + kx^2 + 5x - 27$.

A. 4 C. $\frac{61}{4}$

B. $\frac{4}{61}$ D. 61

18. Find k so that $(x - 2)$ is a factor of $x^3 + kx - 4$.

A. –3 C. –1

B. –2 D. 0

19. Factor $8x^3 - 729$ completely.

A. $(2x - 9)(4x^2 - 18x + 81)$ C. $(2x + 9)(4x^2 + 18x + 81)$

B. $(2x + 9)(4x^2 - 18x + 81)$ D. $(2x - 9)(4x^2 + 18x + 81)$

20. Factor $P(x) = x^4 + x^3 + x^2 + x$.

A. $x(x + 1)(x^2 + 1)$ C. $x(x - 1)(x^2 + 1)$

B. $x(1)(x^2 + 1)$ D. $x(-1)(x^2 + 1)$

21. Below is the solution when $P(x) = (x^3 + 6x^2 + 2x - 12)$ is divided by $(x + 2)$.

$$
\begin{array}{r|rrrr}
-2 & 1 & 6 & 2 & -12 \\
 & & -2 & -8 & 12 \\
\hline
 & 1 & 4 & -6 & 0
\end{array}
$$

Express the third row as a polynomial in x.

A. $x^2 - 4x - 6$
C. $x^2 + 4x + 6$
B. $x^2 - 4x + 6$
D. $x^2 + 4x - 6$

22. If $(7x^4 - 5x^5 - 7x^3 + 2x - 3)$ is divided by $(x + 3)$ using synthetic division, the numbers in the first row would be

A. –5 7 –7 0 2 –3
C. 1 7 –7 0 2 –3
B. –7 –7 –5 0 2 –3
D. –3 7 –7 0 2 –5

23. Given $P(x) = 2x^3 + 3x^2 - 5x - 12$. What is the value of $P(3)$?

A. 56
C. 54
B. 55
D. 53

24. Gabriel used synthetic division to find the quotient if $(5x^2 - 16x + 4x^3 - 3)$ is divided by $(x - 2)$. He obtained –19 as remainder. His solution is shown below.

2	5	–16	4	–3
		10	–12	–16
	5	–6	–8	–19

What is the error?

i. The sign of the divisor was not changed.
ii. The terms of the polynomial were not arranged according to decreasing powers of x.
iii. The sum entries in the third row are incorrect.
iv. The numerical coefficients of the first row were not properly written.

A. i only
C. ii and iv only
B. ii only
D. i and iii only

25. Genber will evaluate an 8th degree polynomial in x at $x = 10$ using the Remainder Theorem and synthetic division. How many coefficients of x will be written in the first row of the synthetic division procedure?

A. 8
C. 10
B. 9
D. 11

26. Which of the following is NOT a root of $x(x + 3)(x + 3)(x - 1)(2x + 1) = 0$?

i. 0 iii. -1

ii. -3 iv. $\frac{1}{2}$

A. i only
B. ii only
C. i and ii only
D. iii and iv only

27. Find a cubic polynomial equation with roots -2, 2, and 4.

A. $x^3 + 4x^2 - 4x + 16 = 0$
B. $x^3 - 4x^2 - x + 16 = 0$
C. $10x^3 - x^2 - x + 16 = 0$
D. $x^3 - 4x^2 - 4x + 16 = 0$

28. How many positive real roots does $x^4 - x^3 - 11x^2 + 9x + 18 = 0$ have?

A. 0
B. 1
C. 2
D. 3

29. One of the roots of the polynomial equation $2x^3 + 9x^2 - 33x + 14 = 0$ is 2. Find the other roots.

A. $\frac{1}{2}$ and 7
B. $-\frac{1}{2}$ and 7
C. $\frac{1}{2}$ and -7
D. $-\frac{1}{2}$ and -7

30. If $P(-2) = 0$, which of the following statements is true about $P(x)$?

A. $x + 2$ is a factor of $P(x)$
B. 2 is root of $P(x) = 0$
C. $P(x) = 0$, has two negative roots
D. $P(0) = -2$

Part II

Read and analyze the situation below. Then, answer the questions or perform the tasks that follow.

Your City Government Projects Office provides guidance and training for local governments, including municipalities or regional mobility authorities in the development of transportation projects. One of its current projects involves the construction of recreational facilities for the city's residents.

Suspose you were one of the engineers of the said project and your job was to renovate/improve the walkway, patio, and driveway. After your ocular inspection, you noticed that a rectangular floor measuring (10 m by 14 m) needed to be fixed. Likewise, your plan is to put brick paves to ensure that the walkway is strong and durable.

Consider the following:

1. Each piece of brick pave is a square with an edge of 50 cm. How many pieces of brick paves will be needed to cover the rectangular floor that needs fixing?
2. If one bag of adhesive cement for brick paves can cover 10 sq. m, how many bags of adhesive cement will be needed?
3. Make a model to illustrate the situation with appropriate mathematical solutions.

Rubric for Rating the Output

Score	Descriptors
4	The problem is properly modelled with appropriate mathematical concepts used in the solution and a correct final answer is obtained.
3	The problem is properly modelled with appropriate mathematical concepts partially used in the solution and a correct final answer is obtained.
2	The problem is not properly modelled, other alternative mathematical concepts are used in the solution, and the correct final answer is obtained.
1	The problem is not properly modelled by the solution presented and the final answer is incorrect.

IV. LEARNING GOALS AND TARGETS

After going through this module, you should be able to demonstrate understanding of key concepts of polynomials and polynomial equations and formulate and solve problems involving these concepts through appropriate and accurate representations.

Lesson 1 Division of Polynomials

What to Know

In this lesson, you will recall operations on polynomials with emphasis on division.

Activity 1: Spot the Difference

Look at each pair of expressions below. Identify the expression that is not a polynomial from each. Give reasons for your answers.

	A	B
1.	$2x + 1$	$\frac{2}{x} + 1$
2.	$x^{-3} + 2x^2 - 7$	$x^3 + 2x^2 - 7$
3.	$2\sqrt{x}$	$x\sqrt{2}$
4.	$2x^3 - 3x^{\frac{1}{2}} + x - 4$	$2x^3 - 3x^2 + x - 4$
5.	$(x + 5)(9x + 1)^2(x - 4)$	$\frac{(x+5)(9x+1)^{-2}}{(x-4)}$

Did this activity help you recall what a polynomial expression is?

A ***polynomial expression*** *P(x)* is an expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0,\ a_n \neq 0$$

where the nonnegative integer n is called the degree of the polynomial and coefficients $a_0, a_1, \ldots, a_n$ are real numbers.

The terms of a polynomial may be written in any order. However, we often follow the convention of writing the terms in decreasing powers of the variable *x.* In this case, the polynomial expression is said to be in ***standard form***.

Activity 2: **Divide and Write**

Divide the following and write an equivalent equation by following the given example.

Example: $19 \div 5 = 3 + \frac{4}{5} \leftrightarrow 19 = 3(5) + 4$

1. $29 \div 5$ = ___________ $\leftrightarrow$ ___________
2. $34 \div 7$ = ___________ $\leftrightarrow$ ___________
3. $145 \div 11$ = ___________ $\leftrightarrow$ ___________
4. $122 \div 7$ = ___________ $\leftrightarrow$ ___________
5. $219 \div 15$ = ___________ $\leftrightarrow$ ___________

> The procedure above can be applied when dividing polynomials. You can see this in the discussion below.

Let us divide $(2x^2 + 5x - 23)$ by $(x + 5)$.

$$
\begin{array}{rrll}
 & 2x - 5 & \leftarrow & \text{Quotient} \\
\text{Divisor} \rightarrow \; x + 5 & \overline{\big)\, 2x^2 + 5x - 23} & \leftarrow & \text{Dividend} \\
 & \underline{2x^2 + 10x } & & \\
 & -5x - 23 & & \\
 & \underline{-5x - 25} & & \\
 & 2 & \leftarrow & \text{Remainder}
\end{array}
$$

You can write the result as follows.

$$\frac{\overbrace{2x^2 + 5x - 23}^{\text{Dividend}}}{\underbrace{x + 5}_{\text{Divisor}}} = \overbrace{2x - 5}^{\text{Quotient}} + \frac{\overset{\text{Remainder}}{\downarrow}}{}\frac{2}{x + 5} \quad \text{or} \quad \overbrace{2x^2 + 5x - 23}^{\text{Dividend}} = \underbrace{(2x - 5)}_{\text{Quotient}}\overbrace{(x + 5)}^{\text{Divisor}} + \underset{\text{Remainder}}{\underset{\uparrow}{2}}$$

In general, if $P(x)$ and $D(x)$ are polynomials with $D(x) \neq 0$, we can write $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$ or $P(x) = Q(x) \bullet D(x) + R(x)$, where $R(x)$ is either 0 or its degree is less than the degree of $D(x)$. If $R(x) = 0$, then we say that $D(x)$ is a factor of $P(x)$.

Dividing Polynomials

As previously shown, the procedure for dividing a polynomial by another polynomial is similar to the procedure used when dividing whole numbers. Another example is shown below.

Example: $(10x^2 + 2x^4 + 8 + 7x^3) \div (2x^2 + x - 1)$

Solution:

First, write the dividend in standard form and insert zeros as coefficients of any missing term to obtain $2x^4 + 7x^3 + 10x^2 + 0x + 8$. Both dividend and divisor should be in standard form. The long division method is shown below.

$$\begin{array}{lrll}
 & & x^2 + 3x + 4 & \leftarrow \text{Quotient} \\
\text{Divisor} \rightarrow & 2x^2 + x - 1 & \overline{)\,2x^4 + 7x^3 + 10x^2 + 0x + 8} & \leftarrow \text{Dividend} \\
 & & \underline{2x^4 + x^3 - x^2} & \text{Subtract} \\
 & & 6x^3 + 11x^2 + 0x & \\
 & & \underline{6x^3 + 3x^2 - 3x} & \text{Subtract} \\
 & & 8x^2 + 3x + 8 & \\
 & & \underline{8x^2 + 4x - 4} & \text{Subtract} \\
 & & -x + 12 & \leftarrow \text{Remainder}
\end{array}$$

Hence, $\frac{2x^4 + 7x^3 + 10x^2 + 8}{2x^2 + x - 1} = x^2 + 3x + 4 + \frac{-x + 12}{2x^2 + x - 1}$.

Activity 3: **Divide and Write it in Form**

Perform the indicated division and write your answers in the form $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$ as shown in the following example;

$$(x^4 + x^2 - 5) \div (x - 5) = \frac{x^4 + x^2 - 5}{x - 5} = x^3 + 5x^2 + 26x + 130 + \frac{645}{x - 5}$$

1. $(5x^2 - 17x - 15) \div (x - 4)$
2. $(6x^3 - 16x^2 + 17x - 6) \div (3x - 2)$
3. $(2x^4 + x^3 - 19x^2 + 18x + 5) \div (2x - 5)$
4. $(4x^5 + 6x^4 + 5x^2 - x - 10) \div (2x^2 + 3)$
5. $(4x^5 - 25x^4 + 40x^3 + 3x^2 - 18x) \div (x^2 - 6x + 9)$

How did you find the activity? What can you say about the procedure?

There is a shorter procedure when a polynomial is to be divided by a binomial of the form $(x - r)$. This method is called ***synthetic division***. In this procedure, we write only the coefficients.

The steps outlined below illustrate synthetic division. The procedure involves writing numbers in three rows.

Example 1. Use synthetic division to divide $P(x) = (3x^3 + 4x^2 + 8)$ by $(x + 2)$.

Step 1: Arrange the coefficients of $P(x)$ in descending powers of x, placing 0s for the missing terms. The leading coefficient of $P(x)$ becomes the first entry of the third row.	$\begin{array}{r\|rrrr} & 3 & 4 & 0 & 8 \\ & & & & \\ \hline & 3 & & & \end{array}$
Step 2: Place the value of r in the upper left corner. In this example, $x - r = x + 2 = x - (-2)$, so $r = -2$.	$\begin{array}{r\|rrrr} -2 & 3 & 4 & 0 & 8 \\ & & & & \\ \hline & 3 & & & \end{array}$

Step 3: The first number in the second row (–6) is the product of r (–2) and the number in the third row (3) of the preceding column. The second number in the third row (–2) is the sum of the two numbers (4 and –6) above it.	$\begin{array}{r\|rrrr} -2 & 3 & 4 & 0 & 8 \\ & & -6 & & \\ \hline & 3 & -2 & & \end{array}$
Step 4: Repeat the procedure described in Step 3 until the last number in the third row is obtained.	$\begin{array}{r\|rrrr} -2 & 3 & 4 & 0 & 8 \\ & & -6 & 4 & -8 \\ \hline & 3 & -2 & 4 & 0 \end{array}$
Step 5: Write the quotient $Q(x)$. Note that the degree of $Q(x)$ is one less than the degree of $P(x)$.The entries in the third row give the coefficients of $Q(x)$ and the remainder R.	$Q(x) = 3x^2 - 2x + 4, \quad \text{R} = 0$

A concise form of Steps 1 to 5 is shown below:

Divisor $(x + 2)$ Dividend $(3x^3 + 4x^2 + 8)$

$$\begin{array}{r|rrrrl} -2 & 3 & 4 & 0 & 8 & \\ & & -6 & 4 & -8 & \\ \hline & 3 & -2 & 4 & 0 & \longleftarrow \text{Remainder: 0} \end{array}$$

Quotient: $3x^2 - 2x + 4$

Example 2. Use synthetic division to find the quotient of

$$(x^4 + 8x^2 - 5x^3 - 2 + 15x) \div (x - 3)$$

Solution:

By inspection, the dividend is not in standard form, so there is a need to rearrange the terms of the polynomial,

Thus, $x^4 + 8x^2 - 5x^3 - 2 + 15x = x^4 - 5x^3 + 8x^2 + 15x - 2$.

Then, write the coefficients of the polynomial in the first row. Follow the steps described in Example 1.

3	1	– 5	8	15	–2
		3	–6	6	63
	1	–2	2	21	61

Therefore, the quotient is ($x^3 - 2x^2 + 2x + 21$) and the remainder is 61.

Example 3. Use synthetic division to find the quotient of $(6x^5 - x^4 - 32x^3 - 20x^2 + 5x + 8) \div (2x - 3)$.

Solution:

Observe that the divisor is not of the form $(x - r)$. However, note that $2x - 3 = 2\left(x - \frac{3}{2}\right)$. Therefore, the problem can be restated as follows:

$$(6x^5 - x^4 - 32x^3 - 20x^2 + 5x + 8) \div 2\left(x - \frac{3}{2}\right)$$

Thus, we first use synthetic division to divide $(6x^5 - x^4 - 32x^3 - 20x^2 + 5x + 8)$ by $\left(x - \frac{3}{2}\right)$, and then divide the result by 2 to get the final answer.

$\frac{3}{2}$	6	–1	–32	–20	5	8
		9	12	–30	–75	–105
	6	8	–20	–50	–70	–97

Now, let us divide the result $\left(6x^4 + 8x^3 - 20x^2 - 50x - 70 + \frac{-97}{x - \frac{3}{2}}\right)$ by 2.

To get the final answer, $3x^4 + 4x^3 - 10x^2 - 25x - 35 + \frac{-97}{2x - 3}$.

Now that you have learned about the division of polynomials, you may try the activities in the next section.

Web Links The following websites provide more information about polynomial division.	http://www.mathsisfun.com/algebra/polynomials-division-long.html http://www.youtube.com/watch?v=qd-T-dTtnX4 http://www.purplemath.com/modules/polydiv2.htm

What to Process

Your goal in this section is to apply the key concepts of dividing polynomials. Use mathematical ideas and examples presented in answering the succeeding activities.

Activity 4: Finding the Divisor, Dividend, and Quotient Using Synthetic Division

Identify the divisor, dividend, and quotient in each item below. Write your answers in your notebook.

1.

1	5	0	3	–8
		5	5	8
	5	5	8	0

Answer: Divisor ________ Dividend ________ Quotient ________

2.

–2	1	5	2	7	30
		–2	–6	8	–30
	1	3	–4	15	0

Answer: Divisor ________ Dividend ________ Quotient ________

3.

3	2	0	0	–54
		6	18	54
	2	6	18	0

Answer: Divisor ________ Dividend ________ Quotient ________

4.

–4	-3	1	0	–208
		12	–52	208
	-3	13	–52	0

Answer: Divisor ________ Dividend ________ Quotient ________

5.

5	2	1	–7	–240
		10	55	240
	2	11	48	0

Answer: Divisor ________ Dividend ________ Quotient ________

This activity helped you identify the quotient in a synthetic division procedure. In the next activity, you will match a division problem with a corresponding solution.

Activity 5: Matching a Polynomial Division Problem with Its Synthetic Counterpart

Match **Column I** with the appropriate synthetic division in **Column II.** Write the letter of the correct answer.

Column I

1. $(2x + x^3 + 7x^2 - 40) \div (x - 2)$
2. $(6x^2 + x^3 + 2x + 44) \div (x + 2)$
3. $(x^3 + 35 + 9x^2 + 13x) \div (x - 5)$
4. $(4x^3 + 26x + 320 + 21x^2) \div (x + 5)$
5. $(-13x + 2x^3 - 5x^2 - 15) \div (2x + 5)$

Column II

A.
$$\begin{array}{r|rrrr} -2 & 1 & 6 & 2 & 44 \\ & & -2 & -8 & 12 \\ \hline & 1 & 4 & -6 & 56 \end{array}$$

B.
$$\begin{array}{r|rrrr} \frac{-5}{2} & 2 & -5 & -13 & -15 \\ & & -5 & 25 & -30 \\ \hline & 2 & -10 & 12 & -45 \end{array}$$

C.
$$\begin{array}{r|rrrr} -5 & 4 & 21 & 26 & 320 \\ & & -20 & -5 & -105 \\ \hline & 4 & 1 & 21 & 215 \end{array}$$

D.
$$\begin{array}{r|rrrr} 5 & 1 & 9 & 13 & 35 \\ & & 5 & 70 & 415 \\ \hline & 1 & 14 & 83 & 450 \end{array}$$

E.
$$\begin{array}{r|rrrr} 2 & 1 & 7 & 2 & -40 \\ & & 2 & 18 & 40 \\ \hline & 1 & 9 & 20 & 0 \end{array}$$

In the next activity, you will perform synthetic division on your own.

Activity 6: Dividing Polynomials Using Synthetic Division

Use synthetic division to find the quotient and remainder in each of the following. Write your complete solutions on a separate sheet of paper.

1. $(3x^3 + x^2 - 22x - 25) \div (x - 2)$ Quotient:________ Remainder:________

2. $(x^3 + 4x^2 - x - 25) \div (x + 5)$ Quotient:________ Remainder:________

3. $(6x^3 - 5x^2 + 4x - 1) \div (3x - 1)$ Quotient:________ Remainder:________

4. $(2x^4 - 9x^3 + 9x^2 + 5x - 1) \div (2x + 1)$ Quotient:________ Remainder:________

5. $(2x^4 + 5x^3 + 3x^2 + 8x + 12) \div (2x + 3)$ Quotient:________ Remainder:________

> Can you now perform synthetic division? In the next activity, not all tasks can be solved easily by synthetic division. Make sure you use long division when necessary.

Activity 7: Dividing Polynomials

Find the quotient and the remainder by using synthetic division. Write your complete solution on a separate sheet of paper.

1. $(x^2 + 3x + 10) \div (x + 2)$ Quotient: ________ Remainder: ________

2. $(10x^3 + 5x^2 + 75x - 40) \div (2x + 1)$ Quotient: ________ Remainder: ________

3. $(12x^3 + 10x^2 + 5x + 1) \div (3x + 1)$ Quotient: ________ Remainder: ________

4. $(3x^4 - x^3 + x - 2) \div (3x^2 + x + 1)$ Quotient:________________
Remainder:______________

5. $(4x^5 - 25x^4 + 40x^3 + 5x^2 - 30x - 18) \div (x^2 - 6x + 9)$
Quotient:________________
Remainder:______________

Activity 8: Apply Your Skills

Solve the following problems. Show your complete solutions.

1. The total cost of $(3a - 2b)$ units of cell phone is $(6a^2 + 5ab - 6b^2)$ pesos. What expression represents the cost of one cell phone?

2. If one ream of bond paper costs $(3x - 4)$ pesos, how many reams can you buy for $(6x^4 - 17x^3 + 24x^2 - 34x + 24)$ pesos?

3. If a car covers $(15x^2 + 7x - 2)$ km in $(3x + 2)$ hours, what is the average speed in km/hr?

4. The volume of a rectangular solid is $(x^3 + 3x^2 + 2x - 5)$ cubic cm, and its height is $(x + 1)$ cm. What is the area of its base?

5. The area of a parallelogram is $(2x^2 + 11x - 9)$ square units. If the length is given by $(2x - 3)$ units, what expression represents its width?

6. If a car moving at a constant rate travels $(2x^3 - x^2 - 4x + 3)$ km in $(x^2 - 2x + 1)$ hours, what is the rate of the car in km per hour?

7. A tailor earns $(12y^2 + y - 35)$ pesos for working $(3y - 5)$ hours. How much does he earn per hour?

What to Reflect on and Understand

Your goal in this section is to take a closer look at some of the ideas in this lesson. The activities will help you assess your understanding of division of polynomials.

Activity 9: **Solve and Express in Polynomial Form**

Answer each of the following completely.

1. If $r = 2x^3 + 4x^2 - x - 6$ and $s = x - 2$. What is $\frac{r}{s}$?

2. Find the quotient when $(x^3 - 6x^2 + 2x + 8)$ is divided by $(x - 3)$.

3. What must be multiplied to $(x^2 + 2x + 1)$ to get $(x^4 + x^3 + x^2 + 3x + 2)$?

4. If a square has a perimeter of $(2x - 48)$ meters, what expression represents its area?

5. Suppose the area of a rectangle is $(6x^2 - 7x + 14)$ square units. If its width is $(2x - 5)$ units, what expression represents its length? How about its perimeter?

After performing each activity, are you now confident about your knowledge of division of polynomials? Try to express your insights through the following activity.

What to Transfer

The next section will help you use division of polynomial to solve some real-world problems.

Activity 10: Solve, then Decide

Solve the following problems. Show your complete solutions.

1. Mr. Aquino wants to paint the ceiling of a room that has a length of $(c^2 + 2cd + d^2)$ meters and a width of $(c + d)$ meters. If one can of paint will cover $(c + d)^2$ square meter, what is the minimum number of cans of paint needed? Express your answer as a polynomial.

2. The side of a square lot is $(5x - 3)$ meters. How many meters of fencing materials are needed to enclose the square lot? If one square meter of the lot costs Php15,000, what is the cost of the square lot?

3. A rectangular garden in a backyard has an area of $(3x^2 + 5x - 6)$ square meters. Its width is $(x + 2)$ meters.

   a. Find the length of the garden.

   b. You decided to partition the garden into two or more smaller congruent gardens. Design a possible model and include mathematical concepts in your design.

Rubric for the Performance Task

CRITERIA	Outstanding 4	Satisfactory 3	Developing 2	Beginning 1	Rating
Accuracy	The computations are accurate. A wise use of key concepts of division of polynomials is evident.	The computations are accurate. Use of key concepts of division of polynomials is evident.	The computations are erroneous and show some use of key concepts of division of polynomials.	The computations are erroneous and do not show some use of key concepts of division of polynomials.	
Stability	The model is well fixed and in place.	The model is firm and stationary.	The model is less firm and show slight movement.	The model is not firm and has the tendency to collapse.	
Creativity	The design is comprehensive and displays the aesthetic aspects of the mathematical concepts learned.	The design is presentable and makes use of the concepts of algebraic representations.	The design makes use of the algebraic representations but not presentable.	The design does not use algebraic representation and not presentable.	
Mathematical Reasoning	The explanation is clear, exhaustive or thorough, and coherent. It includes interesting facts and principles. It uses complex and refined mathematical reasoning.	The explanation is clear and coherent. It covers the important concepts. It uses effective mathematical reasoning.	The explanation is understandable but not logical. It presents only some evidences of mathematical reasoning.	The explanation is incomplete and inconsistent with little evidence of mathematical reasoning.	
				Overall Rating	

Source: D.O. #73, s. 2012

SUMMARY/SYNTHESIS/GENERALIZATION

This lesson was about polynomial division. You learned to:

- divide polynomials using long division;
- determine when synthetic division is appropriate;
- divide polynomials using synthetic division; and
- express the result of division in terms of the quotient and remainder.

Lesson 2 The Remainder Theorem and Factor Theorem

What to Know

In this lesson, you will learn a new method of finding the remainder when a polynomial is divided by $x - r$. You will also learn a method of determining whether or not $x - r$ is a factor of a given polynomial. Before that, you first need to recall your lessons on evaluating polynomials.

Activity 1: Message under the Table

Evaluate the polynomial at the given values of x. Next, determine the letter that matches your answer. When you are done, you will be able to decode the message.

A. $P(x) = x^3 + x^2 + x + 3$

x	-2	-1	0	1	2
$P(x)$					
message					

B. $P(x) = x^4 - 4x^3 - 7x^2 + 22x + 18$

x	-2	-1	0	1	5
$P(x)$					
message					

A. 17	C. -3	E. 5	I. 18
M. 3	N. 78	O. 2	O. 30
P. 6	R. 0	S. -6	T. 23

Guide Questions

1. How did you find the value of a polynomial expression $P(x)$ at a given value of x?
2. What message did you obtain?

Did the activity help you recall how to evaluate a polynomial at the given value? The next activity is a little more challenging.

Activity 2: Equate It

Fill the empty boxes with any of the following terms **$3x^2$, $7x$, $5x$, $3x$, 10, and 8** to satisfy the answer at the end with the given value of x at the beginning. Use each term only once. Use the values at the top to complete the polynomial vertically and the value on the left to complete the polynomial horizontally.

	If $x = -1$		If $x = -2$		If $x = 0$		
If $x = 1$	$5x^3$	–	$2x^2$	+		=	10
	–		+		–		
If $x = 0$	$2x^2$	–		+		=	10
	+		–		+		
If $x = -3$		–		+		=	10
	=		=		=		
	–10		–10		–10		

Guide Questions

1. How did you find the value of a polynomial with the given value of x?
2. What mathematical ideas and skills or strategies did you apply in solving the puzzle game? Why?

Activity 3: **Proving the Remainder Theorem**

Directions: Fill in the blanks with words and symbols that will best complete the statements given below.

Suppose that the polynomial $P(x)$ is divided by $(x - r)$, as follows:

$$\frac{P(x)}{x-r} = Q(x) + \frac{R}{x-r}$$

If $P(x)$ is of degree n, then $Q(x)$ is of degree _____. The remainder R is a constant because ____________________.

Now supply the reasons for each statement in the following table.

STATEMENT	REASON
1. $P(x) = (x - r) \cdot Q(x) + R$	
2. $P(r) = (r - r) \cdot Q(r) + R$	
3. $P(r) = (0) \cdot Q(r) + R$	
4. $P(r) = R$	

The previous activity shows the proof of the Remainder Theorem:

The Remainder Theorem

If the polynomial $P(x)$ is divided by $(x - r)$, the remainder R is a constant and is equal to $P(r)$.

$$\boldsymbol{R = P(r)}$$

Thus, there are two ways to find the remainder when $P(x)$ is divided by $(x - r)$, that is:

(1) use synthetic division, or
(2) calculate $P(r)$.

Similarly, there are two ways to find the value of $P(r)$:

(1) substitute r in the polynomial expression $P(x)$, or
(2) use synthetic division.

Example 1. Find the remainder when $(5x^2 - 2x + 1)$ is divided by $(x + 2)$.

Solution:

a. *Using the Remainder Theorem:*

$P(x) = 5x^2 - 2x + 1$, $r = -2$
$P(-2) = 5(-2)^2 - 2(-2) + 1$
$P(-2) = 5(4) + 4 + 1$
$P(-2) = 20 + 4 + 1 = 25$

Therefore, the remainder when $P(x) = 5x^2 - 2x + 1$ is divided by $x + 2$ is 25.

b. *Using synthetic division:*

$$\begin{array}{r|rrr} -2 & 5 & -2 & 1 \\ & & -10 & 24 \\ \hline & 5 & -12 & 25 \end{array}$$

Thus, the remainder is 25.

Example 2. Find the remainder when $P(x) = 2x^4 + 5x^3 + 2x^2 - 7x - 15$ is divided by $(2x - 3)$.

Solution:

a. *Using the Remainder Theorem:*

Write $2x - 3$ as $2\left(x - \frac{3}{2}\right)$. Here, $r = \frac{3}{2}$.

$$P(x) = 2x^4 + 5x^3 + 2x^2 - 7x - 15$$
$$P\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^4 + 5\left(\frac{3}{2}\right)^3 + 2\left(\frac{3}{2}\right)^2 - 7\left(\frac{3}{2}\right) - 15$$
$$P\left(\frac{3}{2}\right) = 6$$

Thus, $\dfrac{2x^4 + 5x^3 + 2x^2 - 7x - 15}{x - \frac{3}{2}} = 2x^3 + 8x^2 + 14x + 14 + \dfrac{6}{x - \frac{3}{2}}$.

If we divide both sides of the equation by 2, we obtain

$$\frac{2x^4+5x^3+2x^2-7x-15}{2x-3}=x^3+4x^2+7x+7+\frac{6}{2x-3},$$ so the remainder is also 6.

b. *Using synthetic division:*

$$\begin{array}{r|rrrrr} \frac{3}{2} & 2 & 5 & 2 & -7 & -15 \\ & & 3 & 12 & 21 & 21 \\ \hline & 2 & 8 & 14 & 14 & 6 \end{array}$$

Therefore, the remainder is 6.

Sometimes, the remainder when $P(x)$ is divided by $(x - r)$ is 0. This means that $x - r$ is a factor of $P(x)$. Equivalently, $P(r) = 0$. This idea is illustrated by the Factor Theorem.

The Factor Theorem

The polynomial $P(x)$ has $x - r$ as a factor if and only if $P(r) = 0$.

Proof: There are two parts of the proof of the Factor Theorem, namely:
Given a polynomial $P(x)$,

1. If $(x - r)$ is a factor of $P(x)$, then $P(r) = 0$.
2. If $P(r) = 0$, then $(x - r)$ is a factor of $P(x)$.

Activity 4: **Proving the Factor Theorem**

The proof is a consequence of the Remainder Theorem. Fill in the blanks to complete the statement. Write your answers in your notebook.

1. $x - r$ is a factor of $P(x)$ if and only if the remainder R of $P(x) \div (x - r)$ is _______.

2. By the Remainder Theorem, $R = 0$ if and only if ____________.

3. Thus, $(x - r)$ is a factor of $P(x)$ if and only if ____________.

Let us see how the Factor Theorem is used in the following examples.

Example 1. Show that $(x – 1)$ is a factor of $3x^3 – 8x^2 + 3x + 2$.

Solution:

Using the Factor Theorem, we have:

$$\begin{aligned} P(1) &= 3(1)^3 – 8(1)^2 + 3(1) + 2 \\ &= 3 – 8 + 3 + 2 \\ &= 0 \end{aligned}$$

Since $P(1) = 0$, then $x – 1$ is a factor of $3x^3 – 8x^2 + 3x + 2$.

Example 2. Find the value of k for which the binomial $(x + 4)$ is a factor of $x^4 + kx^3 – 4x^2$.

Solution:

If $(x + 4)$ is a factor of $P(x) = x^4 + kx^3 – 4x^2$, we know from the Factor Theorem that $P(–4) = 0$.

$$P(–4) = (–4)^4 + k(–4)^3 – 4(–4)^2 = 0$$
$$256 – 64k – 64 = 0$$
$$\frac{64k}{64} = \frac{192}{64}$$
$$k = 3$$

To check whether the answer is correct or not, use synthetic division to divide $P(x) = x^4 + 3x^3 – 4x^2$ by $x + 4$.

$$\begin{array}{r|rrrrr} & 1 & 3 & -4 & 0 & 0 \\ -4 & & -4 & 4 & 0 & 0 \\ \hline & 1 & -1 & 0 & 0 & 0 \end{array}$$

This shows that the remainder when $P(x)$ is divided by $x + 4$ is 0.

Now that you have learned about the Remainder Theorem and the Factor Theorem for polynomials, you may try the activities in the next section.

What to Process

Practice your skills through the activities in this section.

Activity 5: **Applying the Remainder and Factor Theorems**

Use the Remainder Theorem to find the remainder when the given polynomial is divided by each binomial. Verify your answer using synthetic division. Indicate whether or not each binomial is a factor of the given polynomial.

1. $P(x) = x^3 - 7x + 5$

   a. $x - 1$ b. $x + 1$ c. $x - 2$

2. $\mathrm{P}(x) = 2x^3 - 7x + 3$

   a. $x - 1$ b. $x + 1$ c. $x - 2$

3. $P(x) = 4x^4 - 3x^3 - x^2 + 2x + 1$

   a. $x - 1$ b. $x + 1$ c. $x - 2$

4. $P(x) = 2x^4 - 3x^3 + 4x^2 + 17x + 7$

   a. $2x - 3$ b. $2x + 3$ c. $3x - 2$

5. $P(x) = 8x^4 + 12x^3 - 10x^2 + 3x + 27$

   a. $2x - 3$ b. $2x + 3$ c. $3x - 2$

Activity 6: **Remainder Theorem**

Use the Remainder Theorem to find the remainder R in each of the following.

1. $(x^4 - x^3 + 2) \div (x + 2)$
2. $(x^3 - 2x^2 + x + 6) \div (x - 3)$
3. $(x^4 - 3x^3 + 4x^2 - 6x + 4) \div (x - 2)$
4. $(x^4 - 16x^3 + 18x^2 - 128) \div (x + 2)$
5. $(3x^2 + 5x^3 - 8) \div (x - 4)$

6. $(x^2 - 3x + 7) \div (x + 5)$

7. $(2x^3 - 10x^2 + x - 5) \div (x - 1)$

8. $(x^4 - x^3 + 2) \div (2x + 5)$

9. $(x^3 - x^2 - 8x - 4) \div (3x + 2)$

10. $(x^2 - 8x + 7) \div (5x + 2)$

Guide Questions

1. What is the relation between the remainder and the value of the polynomial at $x = r$ when the polynomial $P(x)$ is divided by a binomial of the form $x - r$?
2. How will you find the remainder when a polynomial in x is divided by a binomial of the form $x - r$?
3. What happens if the remainder is zero?

Activity 7: Factor Theorem

Use the Factor Theorem to determine whether or not the first polynomial is a factor of the second. Then, give the remainder if the second polynomial is divided by the first polynomial.

1. $x - 1$; $x^2 + 2x + 5$
2. $x - 1$; $x^3 - x - 2$
3. $x - 4$; $2x^3 - 9x^2 + 9x - 20$
4. $a - 1$; $a^3 - 2a^2 + a - 2$
5. $y + 3$; $2y^3 + y^2 - 13y + 6$
6. $x - 3$; $-4x^3 + 5x^2 + 8$
7. $b - 2$; $4b^3 - 3b^2 - 8b + 4$
8. $a + 1$; $2a^3 + 5a^2 - 3$

9. $c + 2$; $c^3 + 6c^2 + 3c - 10$

10. $c + 3$; $c^4 - 13c^2 + 36$

In the following activity, one factor of a polynomial is given. Use synthetic division to find the other factor.

Activity 8: Factoring Polynomials

Find the missing factor in each of the following. Write your answers in your notebook.

1. $x^3 - 8 = (x - 2)(\_\_\_\_\_\_\_\_\_\_)$
2. $2x^3 + x^2 - 23x + 20 = (x + 4)(\_\_\_\_\_\_\_\_\_\_)$
3. $3x^3 + 2x^2 - 37x + 12 = (x - 3)(\_\_\_\_\_\_\_\_\_\_)$
4. $x^3 - 2x^2 - x + 2 = (x - 2)(\_\_\_\_\_\_\_\_\_\_)$
5. $2x^3 - x^2 - 2x + 1 = (2x - 1)(\_\_\_\_\_\_\_\_\_\_)$
6. $x^3 - 4x^2 + 4x - 3 = (x - 3)(\_\_\_\_\_\_\_\_\_\_)$
7. $x^3 + 2x^2 - 11x + 20 = (x + 5)(\_\_\_\_\_\_\_\_\_\_)$
8. $3x^3 - 17x^2 + 22x - 60 = (x - 5)(\_\_\_\_\_\_\_\_\_\_)$
9. $4x^3 + 20x^2 - 47x + 12 = (2x - 3)(\_\_\_\_\_\_\_\_\_\_)$
10. $4x^4 - 2x^3 - 4x^2 + 16x - 7 = (2x - 1)(\_\_\_\_\_\_\_\_\_\_)$

What to Reflect on and Understand

This section will require you to apply the Remainder and the Factor Theorems to solve more challenging problems.

Activity 9: Applying the Remainder Theorem

Answer each of the following problems.

1. What is the remainder when $5x^{234} + 2$ is divided by
   a. $x - 1$?
   b. $x + 1$?

2. What is the remainder when $4x^{300} - 3x^{100} - 2x^{25} + 2x^{22} - 4$ is divided by
   a. $x - 1$?
   b. $x + 1$?

3. When divided by $x - 1$, $x + 1$, $x - 2$, and $x + 2$, the polynomial $P(x) = x^4 + rx^3 + sx^2 + tx + u$ leaves a 0 remainder. Find $P(0)$.

4. Determine the value of A so that
   a. $x - 1$ is a factor of $2x^3 + x^2 + 2Ax + 4$.
   b. $x + 1$ is a factor of $x^3 + k^2x^2 - 2Ax - 16$.

5. Use synthetic division to show
   a. $(x + 2)$ and $(3x - 2)$ are factors of $3x^4 - 20x^3 + 80x - 48$.
   b. $(x - 7)$ and $(3x + 5)$ are not factors of $6x^4 - 2x^3 - 80x^2 + 74x - 35$.

At this point you will be given a practical task which will demonstrate your understanding of different concepts you learned from this lesson on polynomials.

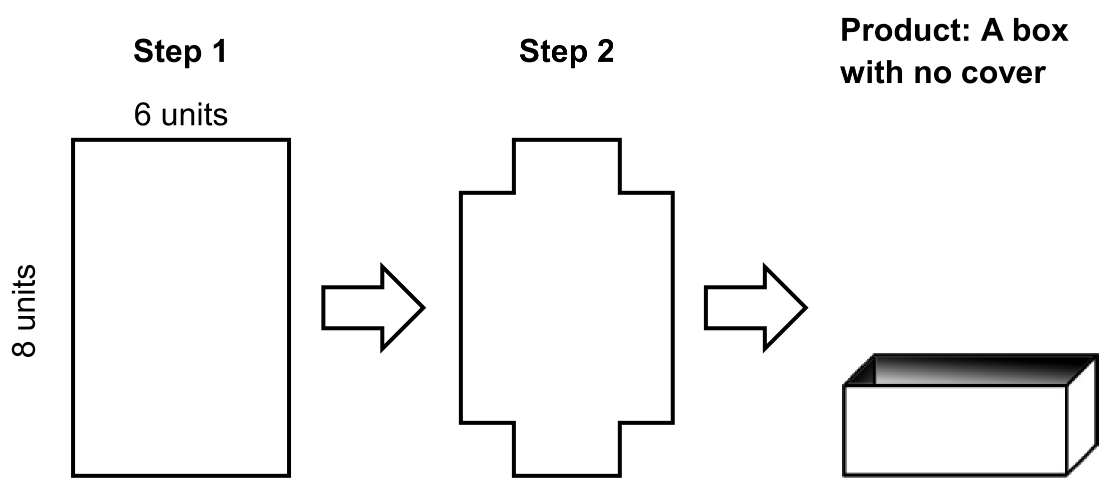
What to Transfer

Polynomial expressions are useful in representing volumes. The next section will help you use division of polynomials as well as the Remainder and the Factor Theorems to solve a real-world problem.

Performance Task

Activity 10: There's a Story Behind a Box

Write a real-life problem based on the procedure shown in the figures below. You may use a situation involving real persons to make the math problem more interesting. You need to consider all significant information in the figures.



Guide Question

Let the situation end with the volume of the resulting box. What insights did you gain from this activity?

Rubric for the Performance Task

CRITERIA	Outstanding 4	Satisfactory 3	Developing 2	Beginning 1	Rating
Accuracy	The computations are accurate and show a wise use of key concepts of division of polynomials.	The computations are accurate and show the use of key concepts of division of polynomials.	The computations are erroneous and show some use of key concepts of division of polynomials.	The computations are erroneous and do not show some use of key concepts of division of polynomials.	
Stability	The model is well- fixed and in place.	The model is firm and stationary.	The model is less firm and show slight movement.	The model is not firm and has the tendency to collapse.	
Creativity	The design is comprehensive and displays the aesthetic aspects of the mathematical concepts learned.	The design is presentable and makes use of the concepts of algebraic representations.	The design makes use of the algebraic representations but not presentable.	The design does not use algebraic representation and it is not presentable.	
Mathematical Reasoning	The explanation is clear, exhaustive or thorough, and coherent. It includes interesting facts and principles. It uses complex and refined mathematical reasoning.	The explanation is clear and coherent. It covers the important concepts. It uses effective mathematical reasoning.	The explanation is understandable but not logical. It contains only some evidences of mathematical reasoning.	The explanation is incomplete and inconsistent, with little evidence of mathematical reasoning.	
				Overall Rating	

Source D.O. #73, s. 2012

SUMMARY/SYNTHESIS/GENERALIZATION

This lesson involved the Remainder and Factor Theorems and their applications. You learned how to:

- find the remainder using synthetic division or the Remainder Theorem;
- evaluate polynomials using substitution or synthetic division; and
- determine whether $(x - r)$ is a factor of a given polynomial.

Lesson 3 Polynomial Equations

What to Know

In Grade 9, you learned how to solve quadratic equations using the Zero-Product Property. In this lesson, you will apply the same property to solve equations involving polynomials in factored form. You will also learn how to factor polynomials and solve general polynomial equations.

Activity 1: Revisiting Roots of Equations

Determine the real root(s) of each equation.

1. $x - 2 = 0$
2. $x + 3 = 0$
3. $x(x - 4) = 0$
4. $(x + 1)(x - 3) = 0$
5. $x^2 + x - 2 = 0$
6. $x^2(x - 9)(2x + 1) = 0$
7. $(x + 4)(x^2 - x + 3) = 0$
8. $2x\,(x^2 - 36) = 0$
9. $(x + 8)(x - 7)(x^2 - 2x + 5) = 0$
10. $(3x + 1)^2(x + 7)(x - 2)^4 = 0$

Guide Questions

1. What do you call the given equations?
2. Describe the roots of an equation.
3. In finding the roots of an equation with degree greater than 1, what have you noticed about the number of roots? Can you recall a principle that supports this?
4. Describe how to solve for the roots of an equation.
5. How many roots does the equation $x^2 + 2x + 1 = 0$ have?

Did you find this activity easy? Did you solve some of these equations mentally? What is the highest degree of the polynomial expressions in the previous activity? Have you encountered equations involving polynomials with a higher degree? The next activity will introduce you to an important principle involving polynomial equations.

Activity 2: Finding the Number of Roots of Polynomial Equations

Some polynomial equations are given below. Complete the table and answer the questions that follow. (If a root occurs twice, count it twice; if thrice, count it three times, and so on. The first one is done for you)

Polynomial Equation	Degree	Real Roots of an Equation	Number of Real Roots
1. $(x+1)^2(x-5)=0$	3	–1 (2 times); 5	3
2. $x-8=0$			
3. $(x+2)(x-2)=0$			
4. $(x-3)(x+1)(x-1)=0$			
5. $x(x-4)(x+5)(x-1)=0$			
6. $(x-1)(x-3)^3=0$			
7. $(x^2-4x+13)(x-5)^3=0$			
8. $(x+1)^5(x-1)^2=0$			
9. $(x^2+4)(x-3)^3=0$			
10. $\left(x-\sqrt{2}\right)^6\left(x+\sqrt{2}\right)^6x^4=0$			

1. Is it easy to give the roots of a polynomial equation when the polynomial is expressed as a product of linear factors? Give a strategy to find roots when a polynomial is expressed as a product of linear factors.

2. What do you observe about the relationship between the number of roots and the degree of a polynomial equation? This relationship was discovered by the German mathematician Karl Friedrich Gauss (1777-1885).

The general statement for the previous observation is known as the Fundamental Theorem of Algebra. We state it here without proof.

Fundamental Theorem of Algebra

If $P(x)$ is a polynomial equation of degree n and with real coefficients, then it has at most n real roots.

3. Consider the following polynomial equations. At most how many real roots does each have?

   a. $x^{20} - 1 = 0$

   b. $x^3 - 2x^2 - 4x + 8 = 0$

   c. $18 + 9x^5 - 11x^2 - x^{23} + x^{34} = 0$

Were you able to find the number of roots of polynomial equations by inspection? The next activity is connected to the problem of finding roots of polynomial equations.

Activity 3: Finding Roots of Polynomial Equations by Applying the Zero-Product Property

Answer the following.

A. When do we say that a real number, say r, is a root of a given polynomial equation in x?

B. Recall the Zero-Product Property. State this property and apply this to solve the equation $(x - 1)(x - 3) = 0$. Is the result consistent with the Fundamental Theorem of Algebra?

C. Find the roots of the following polynomial equations by applying the Zero-Product Property.

1. $(x + 3)(x - 2)(x + 1)(x - 1) = 0$
2. $(x + 5)(x - 5)(x + 5)(x - 1) = 0$
3. $(x + 4)^2(x - 3)^3 = 0$
4. $x (x - 3)^4(x + 6)^2 = 0$
5. $x^2(x - 9) = 0$

D. If a root occurs twice (such as –4 in Item C, Equation 3), the root is called ***a root of multiplicity 2***. In general, if a root occurs n times, it is called ***a root of multiplicity n***. Identify the multiplicity of each root in the equations in Item C.

Now, you are ready to find the roots when the polynomial is not in factored form. The next activity will help you see how.

Activity 4: **Finding Roots of Polynomial Equations by Applying the Factor Theorem**

Answer the following.

A. Let $P(x)$ be a polynomial. Recall the Factor Theorem by completing the statement:

$P(r) = 0$ if and only if $(x - r)$ is ________________.

B. Consider the polynomial equation $x^3 + 6x^2 + 11x + 6 = 0$

Trial 1. Is $x = 1$ a root of the equation?
Using synthetic division,

$$\underline{1\,|} \quad \begin{array}{|cccc} 1 & 6 & 11 & 6 \\ & & & \\ \hline \end{array}$$

The remainder is ___________. Therefore, _________________.

Trial 2. Is $x = -1$ a root of the equation?
Using synthetic division,

$$\begin{array}{r|rrrr} -1 & 1 & 6 & 11 & 6 \\ & & & & \\ \hline \end{array}$$

The remainder is ___________. Therefore, __________________.

The 3rd line of the synthetic division indicates that

$\dfrac{x^3 + 6x^2 + 11x + 6}{x + 1} =$ ___________.

The expression on the right, when equated to zero is called a ***depressed equation*** of the given polynomial equation. The roots of depressed equations are also roots of the given polynomial equation. The roots of this depressed polynomial equation are __________ and __________.

Therefore, the roots of the polynomial equation $x^3 + 6x^2 + 11x + 6 = 0$ are ________, ___________, and ___________.

C. Deepen your skills by discussing the solutions to each polynomial equation with a classmate. As shown above, you first need to guess possible roots of the equation.

1. $x^3 - 2x^2 - x + 2 = 0$

2. $x^3 + 9x^2 + 23x + 15 = 0$

For sure, you have come to a conclusion that it is not always easy to guess the roots of a polynomial equation. A more systematic approach is to limit the roots that one needs to try when solving a polynomial equation. The next activity will demonstrate this.

Activity 5: Exploring the Rational Root Theorem

Complete the table below. Verify that the given numbers in the last column of the table are rational roots of the corresponding polynomial equation.

Polynomial Equation	Leading Coefficient	Constant Term	Roots
1. $x^3 + 6x^2 + 11x - 6 = 0$	1		1, 2, 3
2. $x^3 - x^2 - 10x - 8 = 0$		–8	–2, –1, 4
3. $x^3 + 2x^2 - 23x - 60 = 0$	1		–4, –3, 5
4. $2x^4 - 3x^3 - 4x^2 + 3x + 2 = 0$	2		$-\frac{1}{2}, -1, 1, 2$
5. $3x^4 - 16x^3 + 21x^2 + 4x - 12 = 0$		–12	$-\frac{2}{3}, 1, 2, 3$

Do the task in item 1 below, and answer the questions in items 2 and 3.

1. For each equation, list all possible rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient.
   Example: In Equation 1, $x^3 + 6x^2 + 11x - 6 = 0$, the factors of the constant term –6 are ±6, ±3, ±2, and ±1, and the factors of the leading coefficient 1 are ±1. The rational numbers satisfying the above conditions are $\frac{\pm 6}{\pm 1} = \pm 6$, $\frac{\pm 3}{\pm 1} = \pm 3$, $\frac{\pm 2}{\pm 1} = \pm 2$, and $\frac{\pm 1}{\pm 1} = \pm 1$ (or ± 6, ± 3, ± 2, ± 1). Write a corresponding list for each equation in the table.
2. Look at the roots of each polynomial equation in the table. Are these roots in the list of rational numbers in Question 1?
3. Refer to Equations 1 – 3 in the table. The leading coefficient of each polynomial equation is 1. What do you observe about the roots of each equation in relation to the corresponding constant term?

You may have observed that the leading coefficient and constant term of a polynomial equation are related to the rational roots of the equation. Hence, these can be used to determine the rational solutions to polynomial equations. This observation is formally stated as the Rational Root Theorem, which is the focus of the next activity.

Activity 6: **Proving the Rational Root Theorem**

Based on the previous activity, fill in the blanks below with the correct expressions. Then, complete the proof that follows.

The Rational Root Theorem

Let $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_2x^2 + a_1x + a_0 = 0$, where $a_n \neq 0$ and a_i is an integer for all i, $0 \leq i \leq n$, be a polynomial equation of degree n. If $\frac{p}{q}$, in lowest terms, is a rational root of the equation, then ______ is a factor of a_0 and ______ is a factor of a_n.

Proof:

STATEMENT	REASON
1. $a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + a_{n-2}\left(\frac{p}{q}\right)^{n-2} + \dots a_2\left(\frac{p}{q}\right)^2 + a_1\left(\frac{p}{q}\right) + a_0 = 0$	1. Definition of a root of a polynomial equation.
2.	2. Addition Property of Equality (Add $-a_0$ to both sides).
3.	3. Multiply both sides by q^n.
4.	4. Factor out p on the left side of the equation.
5. Since p is a factor of the left side, then it must also be a factor of the right side.	5. Definition of equality
6. p and q (and hence q^n) do not share any common factor other than ±1.	6. $\frac{p}{q}$ is in lowest terms.
7. p must be a factor of a_0. (This proves the first part of the Rational Root Theorem).	7. p is not a factor of q^n.
8. Similarly, $a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \dots + a_2\left(\frac{p}{q}\right)^2 + a_1\left(\frac{p}{q}\right) + a_0 = -a_n\left(\frac{p}{q}\right)^n$	8.
9.	9. Multiply both sides by q^n.
10.	10. Factor out q on the left side left side of an equation.
11. Since q is a factor of the left side, then it must also be a factor of the right side.	11. Definition of equality
12. q and p (and hence p^n) do not share any common factor other than ±1.	12.
13. q must be a factor of a_n. This proves the second part of the Rational Root Theorem.	13. q is not a factor of p^n.

Now that the Rational Root Theorem has been proved, we are now ready to apply it to solve polynomial equations. Work on the next activity.

Activity 7: **Applying the Rational Root Theorem**

Study the guided solution to the given polynomial equations. Fill in the blanks with appropriate words, numbers, or symbols to complete the solution.

A. Solve $x^3 + x^2 – 12x – 12 = 0$, and write the polynomial in factored form.

Solution:

The equation has at most ______ real roots. The leading coefficient is _____, and its factors are _________ and _________. The constant term is ______, and its factors are ______, ______, ______, ______, _____, _____, _____, _____, _____, _____, ______, and ______. The possible roots of the equation are ± ____, ± ____, ± ____, ± ____, ± ____ and ± ____.

To test if 1 is a root of the given equation, use synthetic division.

1	1	1	–12	–12

Since the remainder is _________, therefore 1 is _________of the equation.

Test if –1 is a root of the equation.

–1	1	1	–12	–12

Since the remainder is _________, therefore -1 is _________of the equation.

Hence, $\dfrac{x^3 + x^2 - 12x - 12}{x+1} = x^2 - 12.$

We can obtain the other roots of $x^3 + x^2 - 12x - 12 = 0$ by solving for the roots of $x^2 – 12 = 0$ by using the quadratic formula.

Its roots are _______ and_________.

To check, simply substitute each of these values to the given equation.

Therefore, the real roots of the polynomial equation $x^3 + x^2 - 12x - 12 = 0$ are ______, ______and______. The factored form of the polynomial $x^3 + x^2 - 12x - 12$ is__________.

B. Now, try to solve the equation given below on your own.

$2x^4 - 11x^3 + 11x^2 - 11x - 9 = 0$

Describe the roots of the equation.

Now that you have gained skill in solving polynomial equations, try to sharpen this skill by working on the next activities.

Web Links The following websites give additional information about the Rational Root Theorem.	https://www.brightstorm.com/math/algebra-2/factoring/rational-roots-theorem/ http://www.youtube,com/watch? v=RXKfaQemtii

What to Process

Practice your skills through the following activities.

Activity 8: Counting the Roots of Polynomial Equations

By inspection, determine the number of real roots of each polynomial equation. Roots of multiplicity *n* are counted *n* times.

1. $(x - 4)(x + 3)^2(x - 1)^3 = 0$
2. $x^2 (x^3 - 1) = 0$
3. $x(x + 3)(x - 6)^2 = 0$
4. $3x (x^3 - 1)^2 = 0$
5. $(x^3 - 8)(x^4 + 1) = 0$

Activity 9: Finding the Roots of Polynomial Equations

Find all real roots of the following equations. Next, write each polynomial on the left side of the equation in factored form. Show your complete solutions.

1. $x^3 - 10x^2 + 32x - 32 = 0$
2. $x^3 - 6x^2 + 11x - 6 = 0$
3. $x^3 - 2x^2 + 4x - 8 = 0$
4. $3x^3 - 19x^2 + 33x - 9 = 0$
5. $x^4 - 5x^2 + 4 = 0$

Activity 10: Completing the List of Roots of Polynomial Equations

One of the roots of the polynomial equation is given. Find the other roots.

1. $-2x^4 + 13x^3 - 21x^2 + 2x + 8 = 0$; $x_1 = -\frac{1}{2}$
2. $x^4 - 3x^2 + 2 = 0$; $x_1 = 1$
3. $x^4 - x^3 - 7x^2 + 13x - 6 = 0$; $x_1 = 1$
4. $x^5 - 5x^4 - 3x^3 + 15x^2 - 4x + 20 = 0$; $x_1 = 2$
5. $2x^4 - 17x^3 + 13x^2 + 53x + 21 = 0$; $x_1 = -1$

How did you find these activities? Did the Rational Root Theorem make it easier for you to find the roots of a polynomial equation? It is important that these ideas are clearly grasped before you proceed to the next activities. Write a mathematical journal that will relate your experience with the Rational Root Theorem.

What to Reflect on and Understand

This activity will broaden your understanding of polynomial equations.

Activity 11: Testing Your Knowledge on Polynomial Equations

Write **TRUE** if the statement is true. Otherwise, modify the underlined word(s) to make it true.

1. The roots of a polynomial equation in x are the values of x that satisfy the equation.
2. Every polynomial equation of degree n has $n - 1$ real roots.

3. The equation $2x^3 - 6x^2 + x - 1 = 0$ has no rational root.

4. The possible roots of $3x^5 - x^4 + 6x^3 - 2x^2 + 8x - 5 = 0$ are $\pm\frac{3}{5}$, ± 3, and ± 5.

5. The only rational root of the equation $x^3 + 6x^2 + 10x + 3 = 0$ is 3.

Activity 12: Exploring Roots of Polynomial Equations

Give 3 examples of polynomial equations with a relatively short list of possible roots, and 3 examples of polynomial equations with a relatively long list of possible roots.

Activity 13: Creating Polynomial Equations

For each item below, give a polynomial equation with integer coefficients that has the following roots.

1. $-1, 3, -6$
2. $\pm 2, \pm 7$
3. $0, -4, -5, \pm 1$
4. $\pm 2, 3, \frac{3}{5}$
5. $\pm 2, -\frac{1}{3}, \frac{2}{7}, 3$

What to Transfer

After going through a number of activities that deepen your understanding of polynomial equations, you are now ready to apply your learning to real-life situations. Work on the next activity.

Activity 14: **Modelling through Polynomial Equations**

Set up a polynomial equation that models each problem below. Then solve the equation, and state the answer to each problem.

1. One dimension of a cube is increased by 1 inch to form a rectangular block. Suppose that the volume of the new block is 150 cubic inches. Find the length of an edge of the original cube.

2. The dimensions of a rectangular metal box are 3 cm, 5 cm, and 8 cm. If the first two dimensions are increased by the same number of centimeters, while the third dimension remains the same, the new volume is 34 cm^3 more than the original volume. What is the new dimension of the enlarged rectangular metal box?

3. The diagonal of a rectangle is 8 m longer than its shorter side. If the area of the rectangle is 60 square m, find its dimensions.

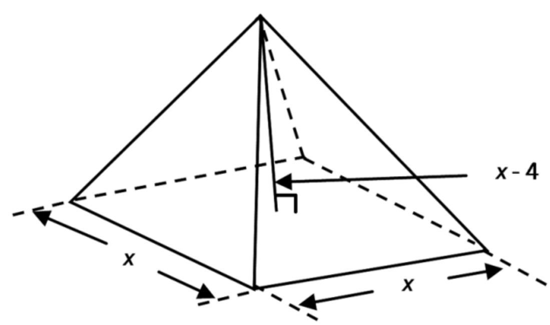
4. Identical squares are cut from each corner of an 8 inch by 11.5 inch rectangular piece of cardboard. The sides are folded up to make a box with no top. If the volume of the resulting box is 63.75 cubic inches, how long is the edge of each square that is cut off?

Activity 15: Using Polynomial Equations to Model Situations

Solve the problem.

Packaging is one important feature in producing quality products. A box designer needs to produce a package for a product in the shape of a pyramid with a square base having a total volume of 200 cubic inches. The height of the package must be 4 inches less than the length of the base. Find the dimensions of the product.

Solution:



Let __________= area of the base

__________= height of the pyramid

If the volume of the pyramid is $V = \frac{1}{3}$(base)(height),

then, the equation that will lead to the solution is 36 = ____________.

The possible roots of the equation are:______________.

Using synthetic division, the roots are_______________.

Therefore, the length of the base of the package is ___________

and its height is _______________________.

SUMMARY/SYNTHESIS/GENERALIZATION

This lesson was about solving polynomial equations and the Rational Root Theorem. You learned how to:

- use the Fundamental Theorem of Algebra to determine the maximum number of real roots of a polynomial;
- solve polynomial equations in factored form;
- solve polynomial equations using the Rational Root Theorem; and
- solve problems that can be modelled by polynomial equations.

GLOSSARY OF TERMS

Degree of a Polynomial - the highest degree of a term in a polynomial

Factor Theorem - the polynomial $P(x)$ has $x - r$ as a factor if and only if $P(r) = 0$

Mathematical Model - a mathematical representation of some phenomena in real world

Polynomial - an algebraic expression of the form $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$, where $a_n \neq 0$, and a_0, a_1, a_2, …, a_n are real numbers

LIST OF THEOREMS USED IN THIS MODULE:

Rational Root Theorem - Let $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0 = 0$ be a polynomial equation of degree n. If $\frac{p}{q}$, in lowest terms, is a rational root of the equation, then p is a factor of a_0 and q is a factor of a_n.

Remainder Theorem - If the polynomial $P(x)$ is divided by $(x - r)$, the remainder R is a constant and is equal to $P(r)$.

Synthetic Division - a short method of dividing polynomial expressions using only the coefficient of the terms

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